Math 2250 Lab 8  Name/Unid: ____________________________

Due Date: 3/06/2014  Class ID: ______ Section: ______

The problems for this week’s lab have been constructed to show how linear algebra concepts from Chapter 4 relate to linear differential equations concepts in Chapter 5.

References: Edwards-Penney Sections 4.3, 4.4, 5.1, 5.2. Course Documents: Slides on Vector Space, Subspace, and Independence; Manuscript on Independence, Span and Basis; Slides on Basis for Linear DE; Slides on Homogeneous and Non-homogeneous Structure, and Superposition; Slides on How to Solve Linear DE

1. Consider the three functions

   \[ y_1(x) = \cos(4x), \quad y_2(x) = \sin(4x), \quad y_3(x) = \sin(4x - \frac{\pi}{6}) \]

   (a) Show that all three functions are particular solutions of the differential equation

   \[ y''(x) + 16y = 0 \]

   (b) The differential equation above is a second order, linear homogeneous DE, so the solution space is two-dimensional. Thus, the three functions \( y_1, y_2, \) and \( y_3 \) above must be linearly dependent. Find a linear dependency. Hint: use a trigonometry angle sum/difference identity.
(c) Explicitly verify that every initial value problem

\[ y'' + 16y = 0 \]
\[ y(0) = a_1 \]
\[ y'(0) = a_2 \]

has a solution of the form

\[ y(x) = c_1 \cos(4x) + c_2 \sin(4x) \]

where \( c_1, c_2 \) are uniquely determined by \( a_1, a_2 \).

Thus, \( \cos(4x) \) and \( \sin(4x) \) form a basis for the two-dimensional solution space of \( y'' + 16y = 0 \).
(d) Find by inspection, particular solutions \( y(x) \) to the non-homogeneous differential equations

\[
\begin{align*}
y''(x) + 16y &= -32 \\
y''(x) + 16y &= 48x
\end{align*}
\]

**Hint:** One particular solution may be a constant; the other, a multiple of \( x \).

(e) Use the principle of superposition together with your work from parts (c) and (d) to find the general solution to the non-homogeneous differential equation

\[
y'' + 16y = -32 + 48x.
\]

(f) Solve the following initial value problem, using your general solution from part (e). Provide a technology answer check.

\[
y'' + 16y = -32 + 48x \\
y(0) = 0 \\
y'(0) = 0
\]
2. Consider the fourth order, homogenous linear differential equation for \( y(x) \)

\[ y^{(4)}(x) = 0 \]

and let \( W \) be the solution space.

(a) Use successive antidifferentiation to find a general solution of this differential equation. Interpret your results using vector space concepts to show that the functions \( y_0(x) = 1, \ y_1(x) = x, \ y_2(x) = x^2, \) and \( y_3(x) = x^3 \) are a basis for \( W \). Thus, the dimension of \( W \) is 4.

(b) Show that the functions \( z_0(x) = 2, \ z_1(x) = x - 4, \ z_2(x) = \frac{1}{2} (x - 4)^2, \) and \( z_3(x) = \frac{1}{6} (x - 4)^3 \) are also a basis for \( W \). **Hint:** If you verify that these four functions satisfy the differential equation and are linearly independent, they will automatically span the four-dimensional solution space and therefore be a basis.

(c) Use a linear combination of the solution basis from part (b) in order to solve the following initial value problem

\[ y^{(4)}(x) = 0 \]
\[ y(4) = 6 \]
\[ y'(4) = 7 \]
\[ y''(4) = 8 \]
\[ y'''(4) = 12 \]

Notice how this basis is adapted to initial value problems at \( x_0 = 4 \) whereas for an initial value problem at \( x_0 = 0 \), the basis in part (a) would have been easier to use.

3. Application of Linear Algebra to Encryption

Any text message can be translated into a string of numbers by assigning to each letter a number, corresponding to its position in the alphabet. For example: A=1, J=10, and Z=26. A blank space between words may be represented by the number 27. An encoding matrix, $E$, may be used to further encrypt a message so that only someone with knowledge of the corresponding decoding matrix, $C$, may understand the message. Encoding and decoding matrices are constructed such that $EC = I$. In other words, the decoding matrix is the inverse of the encoding matrix: $C = E^{-1}$.

(a) Suppose the encoding matrix is

$$ E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix}. $$

Find the decoding matrix $C$.

(b) Decode the following encoded message.

$$ M = \begin{bmatrix} 48 & 42 & 48 & 20 & 46 & 24 & 21 \\ 75 & 64 & 60 & 25 & 73 & 31 & 39 \\ 141 & 113 & 93 & 36 & 128 & 50 & 76 \end{bmatrix}. $$

Observe that since the encoding matrix is a 3x3 matrix, the original message has been split up into seven, 3x1 vectors so the matrix multiplication $CM$ gives the decoded message. This means that the decoded message must be 21 characters long, including blank spaces. For example, suppose I wish to encode the following message: “I STUDY MATH.” Assigning a number to each letter and blank space, this text message is converted to the following string of numbers: “9 27 19 20 21 4 25 27 13 1 20 8.” Given that the encoding matrix is 3x3, I convert this string of numbers into 4, 3x1 vectors

$$ \begin{bmatrix} 9 \\ 27 \\ 19 \\ 20 \\ 25 \\ 27 \\ 1 \end{bmatrix}, \begin{bmatrix} 20 \\ 21 \\ 4 \\ 13 \\ 8 \end{bmatrix}. $$

These vectors may then be augmented into the following matrix

$$ A = \begin{bmatrix} 9 & 20 & 25 & 1 \\ 27 & 21 & 27 & 20 \\ 19 & 4 & 13 & 8 \end{bmatrix}. $$

To encode this message we do the following matrix multiplication
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 4 & 2 \\
\end{pmatrix}
\begin{pmatrix}
9 & 20 & 25 & 1 \\
27 & 21 & 27 & 20 \\
19 & 4 & 13 & 8 \\
\end{pmatrix}
= 
\begin{pmatrix}
55 & 45 & 65 & 29 \\
82 & 66 & 92 & 49 \\
155 & 112 & 159 & 97 \\
\end{pmatrix}.
\]

You may use the decoding matrix you found in part (a), to confirm that you receive the original text message.