Math 2250 Lab 8 Name/Unid: ______________________________

Due Date: 3/06/2014 Class ID: _______ Section: _______

The problems for this week's lab have been constructed to show how linear algebra concepts from Chapter 4 relate to linear differential equations concepts in Chapter 5.

References: Edwards-Penney Sections 4.3, 4.4, 5.1, 5.2. Course Documents: Slides on Vector Space, Subspace, and Independence; Manuscript on Independence, Span and Basis; Slides on Basis for Linear DE; Slides on Homogeneous and Non-homogeneous Structure, and Superposition; Slides on How to Solve Linear DE.

1. Consider the three functions

$$y_1(x) = \cos(4x), \qquad y_2(x) = \sin(4x), \qquad y_3(x) = \sin(4x - \tfrac{\pi}{6})$$

   (a) Show that all three functions are particular solutions of the differential equation

$$y''(x) + 16\,y = 0$$

   .

   (b) The differential equation above is a second order, linear homogeneous DE, so the solution space is two-dimensional. Thus, the three functions y_1, y_2, and y_3 above must be linearly dependent. Find a linear dependency. **Hint:** use a trigonometry angle sum/difference identity.

(c) Explicity verify that every initial value problem

$$y'' + 16\,y = 0$$

$$y(0) = a_1$$
$$y'(0) = a_2$$

has a solution of the form

$$y(x) = c_1\, \cos(4x) + c_2\, \sin(4x)$$

where c_1, c_2 are uniquely determined by a_1, a_2.

Thus, $\cos(4x)$ and $\sin(4x)$ form a basis for the two-dimensional solution space of $y'' + 16\,y = 0$.

(d) Find by inspection, particular solutions y(x) to the non-homogeneous differential equations

$$y''(x) + 16\,y = -32$$

$$y''(x) + 16\,y = 48\,x$$

Hint: One particular solution may be a constant; the other, a multiple of x.

(e) Use the principle of superposition together with your work from parts (c) and (d) to find the general solution to the non-homogeneous differential equation

$$y'' + 16\,y = -32 + 48\,x.$$

(f) Solve the following initial value problem, using your general solution from part (e). Provide a technology answer check.

$$y'' + 16\,y = -32 + 48\,x$$

$$y(0) = 0$$

$$y'(0) = 0$$

.

2. Consider the fourth order, homogenous linear differential equation for y(x)

$$y''''(x) = 0$$

and let W be the solution space.

(a) Use successive antidifferentiation to find a general solution of this differential equation. Interpret your results using vector space concepts to show that the functions $y_0(x) = 1$, $y_1(x) = x$, $y_2(x) = x^2$, and $y_3(x) = x^3$ are a basis for W. Thus, the dimension of W is 4.

(b) Show that the functions $z_0(x) = 2$, $z_1(x) = x - 4$, $z_2(x) = \frac{1}{2}(x-4)^2$, and $z_3(x) = \frac{1}{6}(x-4)^3$ are also a basis for W. **Hint:** If you verify that these four functions satisfy the differential equation and are *linearly independent*, they will automatically *span* the four-dimensional solution space and therefore be a *basis*.

(c) Use a linear combination of the solution basis from part (b) in order to solve the following initial value problem

$$y''''(x) = 0$$
$$y(4) = 6$$
$$y'(4) = 7$$
$$y''(4) = 8$$
$$y'''(4) = 12$$

Notice how this basis is adapted to initial value problems at $x_0 = 4$ whereas for an initial value problem at $x_0 = 0$, the basis in part (a) would have been easier to use.

References: Edwards-Penney Sections 3.4, 3.5. Course Documents: Slides on Matrix Operations; Slides on Inverse Matrices. For more information on applications of matrix operations, particularly Encryption, see Larson & Falvo. (2010). *Elementary Linear Algebra.* Section 2.5: 102-105.

3. **Application of Linear Algebra to Encryption**
Any text message can be translated into a string of numbers by assigning to each letter a number, corresponding to its position in the alphabet. For example: A=1, J=10, and Z=26. A blank space between words may be represented by the number 27. An encoding matrix, E, may be used to further encrypt a message so that only someone with knowledge of the corresponding decoding matrix, C, may understand the message. Encoding and decoding matrices are constructed such that $EC = I$. In other words, the decoding matrix is the inverse of the encoding matrix: $C = E^{-1}$.

   (a) Suppose the encoding matrix is

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix}.$$

   Find the decoding matrix C.

   (b) Decode the following encoded message.

$$M = \begin{bmatrix} 48 & 42 & 48 & 20 & 46 & 24 & 21 \\ 75 & 64 & 60 & 25 & 73 & 31 & 39 \\ 141 & 113 & 93 & 36 & 128 & 50 & 76 \end{bmatrix}.$$

   Observe that since the encoding matrix is a 3x3 matrix, the original message has been split up into seven, 3x1 vectors so the matrix multiplication CM gives the decoded message. This means that the decoded message must be 21 characters long, including blank spaces. For example, suppose I wish to encode the following message: "I STUDY MATH." Assigning a number to each letter and blank space, this text message is converted to the following string of numbers: "9 27 19 20 21 4 25 27 13 1 20 8." Given that the encoding matrix is 3x3, I convert this string of numbers into 4, 3x1 vectors $\begin{pmatrix} 9 \\ 27 \\ 19 \end{pmatrix}, \begin{pmatrix} 20 \\ 21 \\ 4 \end{pmatrix}, \begin{pmatrix} 25 \\ 27 \\ 13 \end{pmatrix}, \begin{pmatrix} 1 \\ 20 \\ 8 \end{pmatrix}$
   These vectors may then be augmented into the following matrix

$$A = \begin{bmatrix} 9 & 20 & 25 & 1 \\ 27 & 21 & 27 & 20 \\ 19 & 4 & 13 & 8 \end{bmatrix}.$$

   To encode this message we do the following matrix multiplication

$$EA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 9 & 20 & 25 & 1 \\ 27 & 21 & 27 & 20 \\ 19 & 4 & 13 & 8 \end{bmatrix} = \begin{bmatrix} 55 & 45 & 65 & 29 \\ 82 & 66 & 92 & 49 \\ 155 & 112 & 159 & 97 \end{bmatrix}.$$

You may use the decoding matrix you found in part (a), to confirm that you receive the original text message.