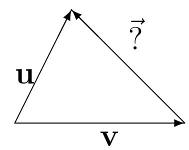
Math 2250 Lab 7 Name/Unid: ______________________________

Due Date: 2/27/2014 Class ID: _______ Section: _______

1. (30 points) Given two vectors $\mathbf{u}, \mathbf{v}$, that are not scalar multiples, i.e $\mathbf{u} \neq c\mathbf{v}$, consider the diagram below. We will assume that the vectors have been labeled so that $|\mathbf{v}| > |\mathbf{u}|$, and that the angle θ between $\mathbf{u}, \mathbf{v}$ is an angle of at most $\frac{\pi}{2}$.



(a) Show $\vec{?} = \mathbf{u} - \mathbf{v}$ using vector addition.

(b) Compute $|\mathbf{u} - \mathbf{v}|^2$ in two ways.

i. by expanding dot product identity $|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$ (which is true because the magnitude squared of any vector is the dot product of that vector with itself).

ii. by using trigonometry and the Pythagorean Theorem on two subtriangles that have right angles - obtained by drawing a line segment perpendicular to $\mathbf{v}$, from the vertex where $\mathbf{u}$ and ? meet, down to $\mathbf{v}$.

(c) Use your work in (b) to deduce that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos\theta$ where θ is the acute angle between $\mathbf{u}$ and $\mathbf{v}$.

References: Edwards-Penney Sections 4.1, 4.6. Course documents: *Slides on Vector Algebra, Linear Algebra Manuscript*, and *Slides on Vector Models and Vector Spaces.* See also: Stewart, J. (2005). *Calculus: Concepts and Contexts*, Sections 9.2, 9.3: Vectors, The Dot Product. (Available in the Math Tutoring Center).

2. (35 points) In the previous problem, we confirmed the well-known identity

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos\theta,$$

where θ is the angle between $\mathbf{u}, \mathbf{v}$.

(a) As a special case of the identity above, explain why $\mathbf{u}$ and $\mathbf{v}$ are perpendicular exactly when $\mathbf{u} \cdot \mathbf{v} = 0$.

(b) Let W be a plane with $(x_0, y_0, z_0) \in W$. Let (x, y, z) be any other point in the plane. Let vector $(a, b, c)^T$ be perpendicular to W (a normal vector). Explain why x, y, z must satisfy the "plane" equation:

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$. Hint: the vector going from (x_0, y_0, z_0) to (x, y, z) must be perpendicular to the normal vector, allowing you to use the dot product identity above.

(c) For the plane through the origin given by $2x + 5y - z = 0$, verify the three position vectors $\mathbf{u} = (1, 0, 2)^T$, $\mathbf{v} = (-3, 2, 4)^T$,$\mathbf{w} = (0, 2, 10)^T$ (from the origin, which is in the plane, to the endpoints of the position vectors, which are also in the plane) are perpendicular to the normal vector $(2, 5, -1)^T$.

(d) In multivariable calculus and physics you learn how to use the cross product of pairs of vectors in 3-space, in order to find perpendicular vectors. Take the cross product of the vectors $\mathbf{u}, \mathbf{v}$ in part (c), and verify that you get a multiple of the normal vector $(2, 5, -1)^T$.

References: Edwards-Penney Sections 4.1, 4.6. Course documents: *Linear Algebra Manuscript*, *Slides on Orthogonality* and *Slides on Vector Models and Vector Spaces.* See also: Stewart, J. (2005). *Calculus: Concepts and Contexts*, Sections 9.4, 9.5: Cross Product, Equations of Lines and Planes. (Available in the Math Tutoring Center).

3. (35 points) (a) Suppose we have a matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ and a square determined by points $(0,0), (0,1), (1,1), (1,0)$. What will the image of the square look like under transformation by the matrix A? i.e.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $(x_1, x_2)^T$ is a point on the original square and $(y_1, y_2)^T$ is its image after transformation by A. Draw the image of the square after this transformation.

(b) Find the area of the parallelogram in (a). (Hint: think of the parallelogram as sitting inside a larger rectangle $0 \leq y_1 \leq 4, 0 \leq y_2 \leq 4$). Compare this area to the determinant of A.

(c) Show, for the general $\mathbf{u}, \mathbf{v}$ in the first quadrant, with $\mathbf{v}$ counterclockwise from $\mathbf{u}$, the area of the parallelgram having the vectors $\mathbf{u}, \mathbf{v}$ as adjacent sides (as in the special case above) always equals $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$.

(d) What happens if you compute $\begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix}$ instead?

References: Edwards-Penney Sections 3.6, 4.1. Course documents: *Linear Algebra Manuscript*, *Determinant Theory Manuscript* and *Slides on Determinants.* See also: Lay, David C. (2003). *Linear Algebra and its Applications*, Section 1.8: Matrix Transformations.