Math 2250 Lab 14 Name/Unid: ______________________________

Due Date: 04/17/2014 Class ID: _______ Section: _______

References: Edwards-Penney: Sections 7.1-7.4, especially Example 2 in Section 7.4. Course Notes: *Systems of DE examples and theory*. Course Slides: *Algebraic Eigenanalysis What is Eigenanalysis?* and **Laplace Second Order Systems**

1. (50 points) Consider a special case of the coupled spring-mass system where three railway cars on a level track are connected by buffer springs that react when compressed, disengaging rather than stretching. As shown in the figure below, suppose the spring constants $k_1 = k_4 = 0$ and $k_2 = k_3 = k$, while the masses of the three railway cars are m_1, m_2, and m_3, respectively.

m_1 m_2 m_3

Assume that the masses slide without friction and that each spring obeys Hooke's Law. That is, the springs extension or compression x and force F are related by $F = -kx$. Furthermore, suppose the displacements x_1, x_2, and x_3 of the three masses (from their respective equilibrium positions) are all positive.

(a) Explain why the application of Newton's Law $F = ma$ to the three masses, m_1, m_2 and m_3, yields the following equations of motion:

$$m_1 x_1''(t) = k_2(x_2(t) - x_1(t))$$
$$m_2 x_2''(t) = -k_2(x_2(t) - x_1(t)) + k_3(x_3(t) - x_2(t))$$
$$m_3 x_3''(t) = -k_3(x_3(t) - x_2(t))$$

(b) Set up the matrix form of the system $\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x}$ where $\mathbf{M}$ represents the mass matrix, $\mathbf{K}$ represents the stiffness matrix, and $\mathbf{x}$ represents the displacement vector $\mathbf{x}= [x_1, x_2, x_3]^T$.

(c) Find the matrix $\mathbf{A}= \mathbf{M^{-1}K}$ where $a_i = \frac{k_i}{m_i}$ for $i = 1, 2, 3$ such that $\mathbf{x}'' = \mathbf{A}\mathbf{x}$.

(d) Given $k_2 = k_3 = k$. Suppose $k = 3600$ lb/ft, $m_1 = m_3 = 400$ slugs and $m_2 = 450$ slugs where a weight of 32 pounds has a mass of 1 slug. Using fps units with mass measured in slugs, find a_1, a_2 and a_3. Show that the coefficient matrix **A** is given by

$$A = \begin{pmatrix} -9 & 9 & 0 \\ 8 & -16 & 8 \\ 0 & 9 & -9 \end{pmatrix}$$

(e) Find the characteristic equation of the coefficient matrix **A** and show that the matrix **A** has eigenvalue-frequency pairs $\lambda_1 = 0$, $\omega_1 = 0$; $\lambda_2 = -9$, $\omega_2 = 3$; and $\lambda_3 = -25$, $\omega_3 = 5$. By an *eigenvalue-frequency pair*, we mean the eigenvalue, λ, with its associated natural frequency ω.

(f) Find the general solution $\mathbf{x}(t)$ using Theorem 1 of Section 7.4 (pg 470) or Laplace resolvent theory as outlined in the course slides **Laplace Second Order Systems**.

(g) Suppose the cars engage at time $t = 0$ with initial positions $x_1(0) = x_2(0) = x_3(0) = 0$ and initial velocities $x_1'(0) = 45$ ft/sec, $x_2'(0) = 0$ ft/sec, and $x_3'(0) = -45$ ft/sec. Show that the railway cars remain engaged until time $t = \frac{\pi}{3}$, after which time they proceed in their respective ways with constant velocities.

(h) The three railway cars travel at constant velocity after $t = \pi/3$ seconds. Find their velocities in ft/sec.

For additional information on epidemiological models, see the *Compartmental models in epidemiology* Wikipedia page.

2. (50 points) Differential equations to describe an epidemic are sometimes given as

$$\frac{dS}{dt} = cI - bSI$$
$$\frac{dI}{dt} = bSI - cI$$

where S measures the number of susceptible people, I represents the number of infected people. Individuals become infected (move from the S class into the I class) at a rate proportional to the product of the number of infected individuals with the number of susceptible people. Individuals recover (move from the I class into the S class) at a rate proportional to the number of infected individuals.

(a) What biological process does each term on the right-hand side of each equation describe?

(b) Explain at least one assumption of this model. For example, as given above, the model assumes only two states: individuals are either infected or susceptible. In other words, once an individual is "recovered" or no longer infected, he or she is once again susceptible to the disease.

(c) Suppose there is a source of mortality, whereby susceptible individuals die at per capita rate k while infected individuals die at a per capita rate that is twice as large. Write a new set of differential equations that incorporates this assumption.

(d) Suppose all individuals give birth at rate b. Moreover, suppose the offspring of susceptible individuals are susceptible while the offspring of infected individuals are infected. Write a new set of differential equations from the given system that incorporates this assumption.

(e) Again, suppose all individuals give birth at rate b but that all offspring are susceptible. How does the system you wrote in part (d) change based on this change in assumption?

(f) Suppose that individuals who leave the infected class through recovery become permanently immune rather than becoming susceptible again. That is, there is a new state for permanently "recovered" individuals. Write a new set of differential equations that incorporates this assumption.