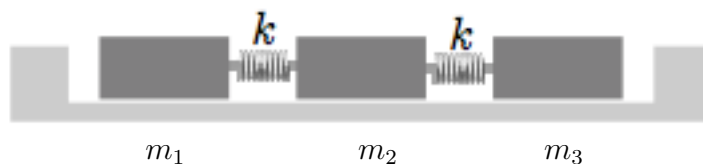


References: Edwards-Penney: Sections 7.1-7.4, especially Example 2 in Section 7.4.
 Course Notes: *Systems of DE examples and theory*. Course Slides: *Algebraic Eigenanalysis What is Eigenanalysis?* and **Laplace Second Order Systems**

1. (50 points) Consider a special case of the coupled spring-mass system where three railway cars on a level track are connected by buffer springs that react when compressed, disengaging rather than stretching. As shown in the figure below, suppose the spring constants $k_1 = k_4 = 0$ and $k_2 = k_3 = k$, while the masses of the three railway cars are m_1, m_2 , and m_3 , respectively.



Assume that the masses slide without friction and that each spring obeys Hooke's Law. That is, the springs extension or compression x and force F are related by $F = -kx$. Furthermore, suppose the displacements x_1, x_2 , and x_3 of the three masses (from their respective equilibrium positions) are all positive.

- (a) Explain why the application of Newton's Law $F = ma$ to the three masses, m_1, m_2 and m_3 , yields the following equations of motion:

$$\begin{aligned} m_1 x_1''(t) &= k_2(x_2(t) - x_1(t)) \\ m_2 x_2''(t) &= -k_2(x_2(t) - x_1(t)) + k_3(x_3(t) - x_2(t)) \\ m_3 x_3''(t) &= -k_3(x_3(t) - x_2(t)) \end{aligned}$$

- (b) Set up the matrix form of the system $\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x}$ where \mathbf{M} represents the mass matrix, \mathbf{K} represents the stiffness matrix, and \mathbf{x} represents the displacement vector $\mathbf{x} = [x_1, x_2, x_3]^T$.
- (c) Find the matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ where $a_i = \frac{k_i}{m_i}$ for $i = 1, 2, 3$ such that $\mathbf{x}'' = \mathbf{A}\mathbf{x}$.

- (d) Given $k_2 = k_3 = k$. Suppose $k = 3600$ lb/ft, $m_1 = m_3 = 400$ slugs and $m_2 = 450$ slugs where a weight of 32 pounds has a mass of 1 slug. Using fps units with mass measured in slugs, find a_1 , a_2 and a_3 . Show that the coefficient matrix \mathbf{A} is given by

$$A = \begin{pmatrix} -9 & 9 & 0 \\ 8 & -16 & 8 \\ 0 & 9 & -9 \end{pmatrix}$$

- (e) Find the characteristic equation of the coefficient matrix \mathbf{A} and show that the matrix \mathbf{A} has eigenvalue-frequency pairs $\lambda_1 = 0$, $\omega_1 = 0$; $\lambda_2 = -9$, $\omega_2 = 3$; and $\lambda_3 = -25$, $\omega_3 = 5$. By an *eigenvalue-frequency pair*, we mean the eigenvalue, λ , with its associated natural frequency ω .
- (f) Find the general solution $\mathbf{x}(t)$ using Theorem 1 of Section 7.4 (pg 470) or Laplace resolvent theory as outlined in the course slides **Laplace Second Order Systems**.
- (g) Suppose the cars engage at time $t = 0$ with initial positions $x_1(0) = x_2(0) = x_3(0) = 0$ and initial velocities $x'_1(0) = 45$ ft/sec, $x'_2(0) = 0$ ft/sec, and $x'_3(0) = -45$ ft/sec. Show that the railway cars remain engaged until time $t = \frac{\pi}{3}$, after which time they proceed in their respective ways with constant velocities.
- (h) The three railway cars travel at constant velocity after $t = \pi/3$ seconds. Find their velocities in ft/sec.

For additional information on epidemiological models, see the *Compartmental models in epidemiology* Wikipedia page.

2. (50 points) Differential equations to describe an epidemic are sometimes given as

$$\begin{aligned}\frac{dS}{dt} &= cI - bSI \\ \frac{dI}{dt} &= bSI - cI\end{aligned}$$

where S measures the number of susceptible people, I represents the number of infected people. Individuals become infected (move from the S class into the I class) at a rate proportional to the product of the number of infected individuals with the number of susceptible people. Individuals recover (move from the I class into the S class) at a rate proportional to the number of infected individuals.

- (a) What biological process does each term on the right-hand side of each equation describe?
- (b) Explain at least one assumption of this model. For example, as given above, the model assumes only two states: individuals are either infected or susceptible. In other words, once an individual is “recovered” or no longer infected, he or she is once again susceptible to the disease.
- (c) Suppose there is a source of mortality, whereby susceptible individuals die at per capita rate k while infected individuals die at a per capita rate that is twice as large. Write a new set of differential equations that incorporates this assumption.
- (d) Suppose all individuals give birth at rate b . Moreover, suppose the offspring of susceptible individuals are susceptible while the offspring of infected individuals are infected. Write a new set of differential equations from the given system that incorporates this assumption.
- (e) Again, suppose all individuals give birth at rate b but that all offspring are susceptible. How does the system you wrote in part (d) change based on this change in assumption?
- (f) Suppose that individuals who leave the infected class through recovery become permanently immune rather than becoming susceptible again. That is, there is a new state for permanently “recovered” individuals. Write a new set of differential equations that incorporates this assumption.

