Math 2250 Lab 12 Name/Unid: ______________________________

Due Date: 04/03/2014 Class ID: _______ Section: _______

References: Edwards-Penney Sections 5.4, 5.6, 10.1-10.5. Course slides: *Basic Laplace Theory*, *Laplace Rules*, *Laplace Table Proofs*, , *Piecewise Functions* and *Electrical Circuits*, especially the Electrical-Mechanical Analogy slide. See also: *Laplace Theory Manuscript*

1. Suppose a certain spring-mass system with no damping satisfies the initial value problem

$$x''(t) + 9x(t) = f(t)$$

$$x(0) = 0, x'(0) = 0$$

where

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ (t-4)/4 & 4 \le t \le 8 \\ 1 & t > 8 \end{cases}$$

(a) The shape of the forcing function $f(t)$ is similar to the head load/unload geometry in a computer hard disk drive. The three intervals correspond to the disk, the unloading ramp and the detent position of the read head. Sketch the graph of f to see why.

(b) Find the general form of the solution in the following cases.

Case $t < 4$.

Case $t > 8$.

(c) Show the details required to rewrite $f(t)$ in terms of the unit step functions $u(t-4)$ and $u(t-8)$, as the expression

$$f(t) = \frac{u(t-4)(t-4) - u(t-8)(t-8)}{4}$$

(d) Solve the initial value problem using Laplace's method. Include a technology answer check.

(e) Plot the solution found in part (d) and find the amplitude of the steady-state solution.

2. This problem illustrates how the response of a mechanical or electrical system to different inputs (forcing functions) provides information about the defining parameters of the system.

   Consider a forced oscillator differential equation for $x(t)$:

   $$ax''(t) + bx'(t) + cx(t) = f(t)$$

   Assume the parameters a, b, c are initially unknown. The input function $f(t)$ is considered to be a variable at our disposal, used to find the values of a, b, c.

   An experiment is performed with $f(t)$ equal to a unit impulse at $t = 0$. An approximation will be used, like a hammer hit of impulse 1, but the experiment is defined by the idealized problem

   $$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.$$

   The output $h(t)$ is fit to a curve, which we assume for exposition purposes is

   $$h(t) = e^{-4t}\sin(2t)u(t) = \begin{cases} e^{-4t}\sin(2t) & t > 0, \\ 0 & t < 0, \end{cases},$$

   called the **unit impulse response** of the system.

   The fundamental result of Laplace convolution theory is

   $$x(t) = \int_0^t h(s)f(t-s)ds, \quad t > 0,$$

   for any forcing function $f(t)$. The value of $x(t)$ depends on the values of the forcing function $f(s)$ for $0 \le s \le t$, and on the value of the unit impulse response $h(s) = e^{-4s}\sin(2s)$, also for the previous times $0 \le s \le t$.

   (a) The Laplace transform $H(s)$ of the impulse response $h(t) = e^{-4t}\sin(2t)$ is called the **transfer function**. What is the formula for $H(s)$?

   (b) Since $x(t)$ is given by the convolution integral of h and f, its Laplace transform $X(s)$ must be the product $H(s)F(s)$. On the other hand, since $x(t)$ solves the forced oscillation initial value problem, you know how to find its Laplace transform $X(s)$ by transforming the differential equation. Equate these two formulas for $X(s)$ to deduce the parameter values a, b, c in the inhomogeneous ODE describing the mechanical or electrical system.

   (c) If this was a mass-spring-dashpot system, what would be the values of m, c, k?
   If it was an electrical system, what would be the values of L, R, C?

3. Consider an RLC Circuit with voltage source $E_0 = 45$ controlled by a switch. Suppose the voltage source is initially turned off. That is, at $t = 0$, $Q(0) = I(0) = 0$. At $t = 30$ seconds, the switch is opened and left open. Let $R = 35\ \Omega$, $L = 0.5$ H, $C = 0.002$ F. This circuit can be modeled as:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) = \begin{cases} 45 & t \geq 30, \\ 0 & t < 30. \end{cases}$$
$$Q(0) = Q'(0) = 0.$$

(a) The Dirac ideal impulse $\delta(t-a)$ is known to satisfy for all continuous functions f the identity

$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

Use the identity with $f(t) = e^{-st}$ to find $\mathcal{L}\{\delta(t-a)\}$.

(b) Find the current equation for the circuit by formal differentiation of the charge equation above. It will contain the formal derivative of a unit step, which is a unit Dirac impulse. Then determine the initial conditions $I(0), I'(0)$.

(c) Find the resulting current I(t) in the circuit using Laplace's method.