1. Using Laplace Transforms to Solve a Linear Second Order System

Consider a coupled mass-and-spring system as depicted in the figure below and described by the system

\[
\begin{align*}
4x'' + 16x - 4y &= 0, \\
2y'' - 4x + 6y &= 12 \sin 2t
\end{align*}
\]

![A Coupled Spring-Mass System](image)

Symbols \(k_1, k_2, k_3\) are Hooke’s constants. Symbols \(m_1, m_2\) are masses. Symbol \(F\) is a force acting on mass \(m_2\) with magnitude \(f(t) = 12 \sin 2t\).

Assumed are values \(m_1 = 4, m_2 = 2, k_1 = 12, k_2 = 4, k_3 = 2\).

The purpose of this project is to solve the system for \(x(t), y(t)\) when at \(t = 0\) both masses are at rest and in the equilibrium position. The external force \(f(t) = 12 \sin 2t\) is applied to the second mass \(m_2\) starting at time \(t = 0\).

(a) Define \(X(s) = \mathcal{L}\{x(t)\}\) and \(Y(s) = \mathcal{L}\{y(t)\}\). Display the Laplace Method details which transform the differential equations and initial conditions \((x(0) = y(0) = x'(0) = y'(0) = 0)\) into the 2 \(\times\) 2 system of linear algebraic equations

\[
\begin{align*}
(s^2 + 4)X(s) - Y(s) &= 0, \\
(s^2 + 3)Y(s) - 2X(s) &= \frac{12}{s^2 + 4}
\end{align*}
\]

Suggestion: Use the corollary Transforms of Higher Derivatives in Section 10.2.

(b) Solve the transformed system of part (a) for Laplace transforms \(X(s)\) and \(Y(s)\), using Cramer’s Rule for 2 \(\times\) 2 systems of linear algebraic equations.

(c) Find the solution \(x(t), y(t)\) by backward table methods and partial fractions. Equivalently, compute \(x(t) = \mathcal{L}^{-1}\{X(S)\}\) and \(y(t) = \mathcal{L}^{-1}\{Y(S)\}\). A technology answer check is expected.