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Differential Equations and Linear Algebra 2250

Midterm Exam 1

Version 1, 14 Feb 2014

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [30%] Solve $y' = \dfrac{x^2 - 2x - 3}{1 + x}$.

   (b) [30%] Solve $y' = \dfrac{\sin(x)}{\cos^2(x) + 1}$.

   (c) [40%] Let $W(t) = e^{3t}$. Find the velocity $v(t)$ from the velocity model

$$\frac{d}{dt}\left(W(t)v(t)\right) = 80e^{-t} + 150e^{-2t}, \quad v(0) = -50$$

and the position $x(t)$ from the position model

$$\frac{dx}{dt} = v(t), \quad x(0) = 111.$$

(a) $y' = (x-3)(x+1)/(1+x) = x - 3$

$y = \frac{x^2}{2} - 3x + c$

(b) $y' = \frac{-du}{u^2+1}$ where $u = \cos(x)$

$y = -\tan^{-1}(\cos(x)) + c$

(c) Quadrature on v-equation

$Wv = -80e^{-t} - 75e^{-2t} + c_1 \Rightarrow v = -80e^{-4t} - 75e^{-5t} + c_1e^{-3t}$

Divide by $W = e^{3t}$

$v(0) = -50 \Rightarrow -50 = -80 - 75 + c_1$ or $c_1 = 105$

$x = \int v = \int (105e^{-3t} - 80e^{-4t} - 75e^{-5t})\,dt$

$x = -35e^{-3t} + 20e^{-4t} + 15e^{-5t} + c_2$

$111 = -35 + 20 + 15 + c_2$ (from $x(0) = 111$) $\Rightarrow c_2 = 111$

$$\boxed{\begin{aligned} v &= 105e^{-3t} - 80e^{-4t} - 75e^{-5t} \\ x &= -35e^{-3t} + 20e^{-4t} + 15e^{-5t} + 111 \end{aligned}}$$

Use this page to start your solution. Attach extra pages as needed.

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided functions F and G exist such that $f(x, y) = F(x)G(y)$.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

$\boxed{X}$ $y' + 2xy = 3y$	$\boxed{X}$ $y' = 3xy^2 + (-xy + x^2y)y$
$\boxed{}$ $y' = \tan(x) + y$	$\boxed{X}$ $e^{x+y}y' = xe^{2y} + (xe^y)^2$

(b) [10%] Give an example $y' = f(x, y)$ which is not separable, not quadrature, but it is linear.

(c) [20%] Apply a classification test to show that $y' + \tan(x)y = -1 - x + \pi y$ is a linear differential equation. Supply all details, including a statement of the test.

(d) [30%] Apply a classification test to show that $y' = y\sec(x + y)$ is not separable. Supply all details, including a statement of the test.

Eq1 (a) $y' = 3y - 2xy$, $y' = (3-2x)y$ Sep

Eq2 (a) $y' = 3xy^2 - xy^2 + x^2y^2 = (3x - x + x^2)y^2$ Sep

Eq3 (a) $y' = \tan(x) + y$ $\quad \dfrac{f_y}{f} = \dfrac{1}{\tan x + y}$ at $y=0$ equals $\dfrac{1}{\tan(x)}$, Depends on x Not Sep

Eq4 (a) $e^x e^y y' = xe^{2y} + x^2e^{2y}$ or $y' = \left(\dfrac{x+x^2}{e^x}\right)e^y$ Sep

(b) $y' = \tan(x) + y$ From Eq 3(a) $\quad \dfrac{\partial f}{\partial y} = 1 \neq 0 \Rightarrow$ Not Quadrature, but linear

(c) Test: $y' = f(x,y)$ is linear provided $\dfrac{\partial f}{\partial y}$ is indep of y

Apply: $f = -\tan(x)y - 1 - x + \pi y$, $\dfrac{\partial f}{\partial y} = -\tan(x) + \pi$

Indep of $y \Rightarrow$ linear.

(d) Test: $y' = f(x,y)$ not separable provided $\dfrac{\partial f / \partial y}{f}$ depends on x

Apply: $f = y\sec(x+y)$, $\dfrac{\partial f}{\partial y} = \sec(x+y) + y\sec(x+y)\tan(x+y)$

$\dfrac{\partial f / \partial y}{y} = \dfrac{y}{1 + y\tan(x+y)}$ at $y = 1$, get $\dfrac{1}{1+\tan(x+1)}$ depends on x

Use this page to start your solution. Attach extra pages as needed.

Then $y' = y\sec(x+y)$ is not separable.

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3. (Solve a Separable Equation)

Given $(xy + 2x + y + 2)y' = \left((1 + x)\sin^2(x)\cos(x) + 2x^2\right)(y^2 + 4y + 4)(y + 3)$.

(a) [80%] Find a non-constant solution in implicit form.
To save time, **do not solve** for y explicitly. No answer check is expected.

(b) [20%] Find all constant solutions, which have the form $y = c$, also called equilibrium solutions.
No answer check is expected.

(a) $(x+1)(y+2)y' = ((1+x)\sin^2 x \cos x + 2x^2)(y^2+4y+4)(y+3)$

$$y' = \left(\sin^2 x \cos x + \frac{2x^2}{1+x}\right)(y+2)(y+3)$$

$$\frac{y'}{(y+2)(y+3)} = \sin^2 x \cos x + \frac{2x^2}{1+x}$$

$$\int (\text{RHS})\,dx = \frac{\sin^3 x}{3} + \int \frac{2(u-1)^2}{u}\,du \quad \text{where } u = 1+x$$

$$= \frac{\sin^3 x}{3} + \int\left(\frac{2u^2}{u} - \frac{4u}{u} + \frac{2}{u}\right)du$$

$$= \frac{\sin^3 x}{3} + u^2 - 4u + 2\ln|u| + c_1$$

$$= \frac{\sin^3 x}{3} + (1+x)^2 - 4(x+1) + 2\ln|x+1| + c_1$$

we'll aborb -4 into c_1 later

$$\int(\text{LHS})\,dx = \int \frac{du}{(u+2)(u+3)} \quad \text{where } u = y(x),\ du = y'(x)\,dx$$

$$= \int\left(\frac{1}{u+2} + \frac{-1}{u+3}\right)du$$

$$= \ln|y+2| - \ln|y+3| + c_2$$

Answer (a): $\ln|y+2| - \ln|y+3| = \frac{\sin^3 x}{3} + x^2 + 2x - 4x + (1-4) + 2\ln|1+x| + C$

Can absorb into C

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Answer (b): $y = -2$ and $y = -3$
To find them, set $y' = 0$ in (a) and solve for y.

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4. (Linear Equations)

(a) [50%] Solve the linear model $2x'(t) = -g + \dfrac{12}{2t+1}x(t)$, $x(0) = 0.125g$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $3x^2\dfrac{dy}{dx} - 2y = 0$.

(c) [30%] Solve $30\dfrac{dy}{dx} = 3 - 5y$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' - \dfrac{6}{2t+1}x = -\dfrac{g}{2}$

$\dfrac{(Wx)'}{W} = -\dfrac{g}{2}$ where $W = e^{\int \frac{-6\,dt}{2t+1}} = e^{\ln|2t+1|^{-3}}$

Let $W = (2t+1)^{-3}$

$(Wx)' = -\dfrac{g}{2}W$ Cross-multiply

$Wx = -\dfrac{g}{2}\int (2t+1)^{-3}dt$ Quadrature method

$Wx = \dfrac{g}{8}(2t+1)^{-2} + c_1$

$x = \dfrac{g}{8}(2t+1) + c_1(2t+1)^3$ Divide by W

$x(0) = 0.125g = \dfrac{g}{8}$, also $x(0) = \dfrac{g}{8} + c_1$

So $c_1 = 0$ and $\boxed{x(t) = \dfrac{g}{8}(2t+1)}$

(b) $y = \dfrac{\text{constant}}{\text{integrating factor}} = \dfrac{c}{e^{\int \frac{-2}{3x^2}dx}} = \boxed{\dfrac{c}{e^{\frac{2}{3x}}}}$

(c) $y_p = \dfrac{3}{5}$

$y_h = \dfrac{c}{e^{\int \frac{5}{30}dx}} = \dfrac{c}{e^{x/6}} = c\,e^{-x/6}$

$y = y_p + y_p = \boxed{\dfrac{3}{5} + c\,e^{-x/6}}$

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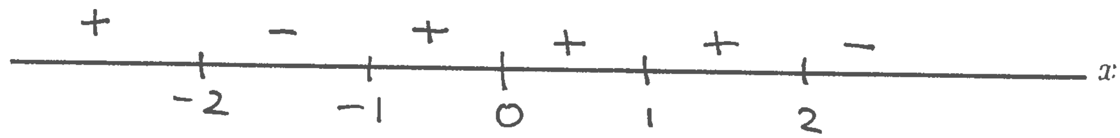
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = e^x \left(3 - |2x - 1|\right)(-2 + x)(x^2 - 4)(x - x^2)^2.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.

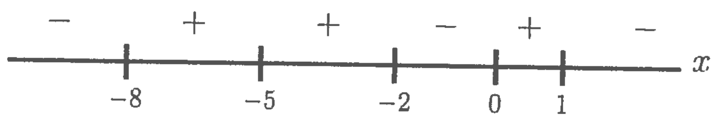


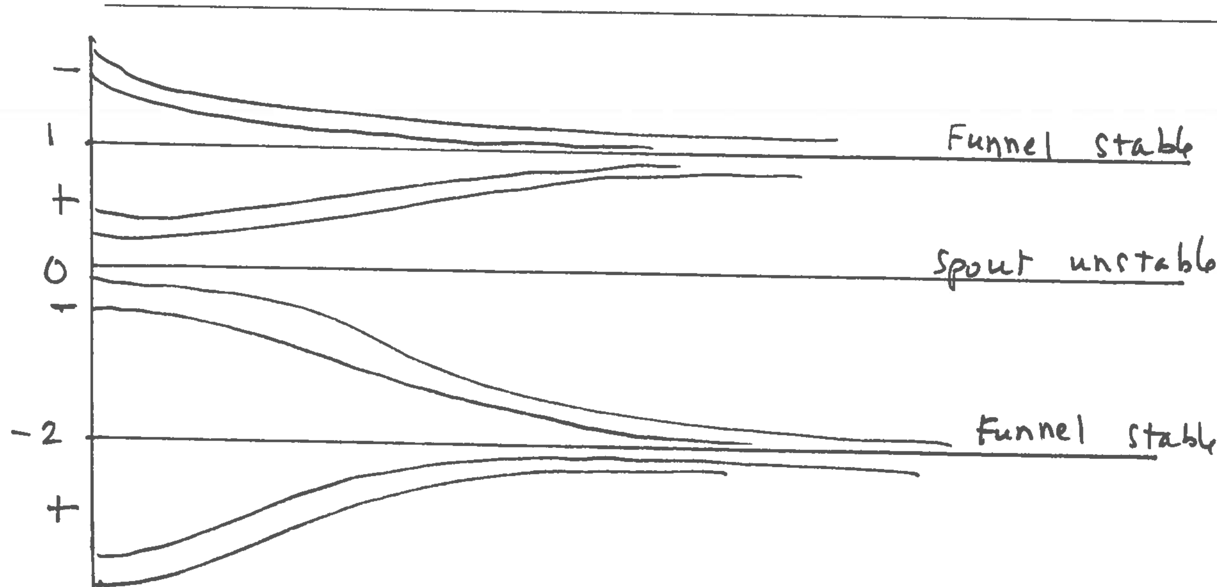
$3 - |2x-1| = 0$
at $x = -1, 2$

$f(x) = e^x \left(3 - |2x-1|\right)(x-2)^2(x+2)\,x^2(1-x)^2$

$g(x) = (3 - |2x-1|)(x+2)$ Determines the sign: test $-3, -1.5, -.5, .5, 1.5, 3$

(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel, spout, node** [neither spout nor funnel], **stable, unstable**.





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