Differential Equations and Linear Algebra 2250
Midterm Exam 1
Version 1, 14 Feb 2014

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)
   (a) [30%] Solve \( y' = \frac{x^2 - 2x - 3}{1 + x} \).
   \[ y = \frac{x^2}{2} - 3x + C \]
   (b) [30%] Solve \( y' = \frac{\sin(x)}{\cos^2(x) + 1} \).
   \[ y = -\tan^{-1}(\cos(x)) + C \]
   (c) [40%] Let \( W(t) = e^{2t} \). Find the velocity \( v(t) \) from the velocity model
   \[ \frac{d}{dt}(W(t)v(t)) = 80e^{-t} + 150e^{-2t}, \quad v(0) = -50 \]
   and the position \( x(t) \) from the position model
   \[ \frac{dx}{dt} = v(t), \quad x(0) = 111 \]
   \[ \begin{align*}
     (a) \quad y' &= (x - 3)(x + 1)/(1 + x) = x - 3 \\
     y &= \frac{x^2}{2} - 3x + C \\
     \quad (b) \quad y' &= -\frac{du}{u^2 + 1} \quad \text{where } u = \cos(x) \\
     y &= -\tan^{-1}(\cos(x)) + C \\
     (c) \quad \text{Quadrature on } v-\text{equation} \\
     Wv &= -80e^{-t} - 75e^{-2t} + c_1 \Rightarrow v = -80e^{-4t} - 75e^{-5t} + c_1 e^{3t} \\
     \text{Divide by } W = e^{3t} \\
     v(0) &= -50 \Rightarrow -50 = -80 - 75 + c_1 \quad \text{or } c_1 = 105 \\
     x &= \int v \, dt = \int (105e^{-3t} - 80e^{-4t} - 75e^{-5t}) \, dt \\
     x &= -35e^{-3t} + 20e^{-4t} + 15e^{-5t} + c_2 \\
     111 &= -35 + 20 + 15 + c_2 \quad \text{from } x(0) = 111 \Rightarrow c_2 = 111 \\
     v &= 105e^{-3t} - 80e^{-4t} - 75e^{-5t} \\
     x &= -35e^{-3t} + 20e^{-4t} + 15e^{-5t} + 111 \\
   \end{align*} \]
   Use this page to start your solution. Attach extra pages as needed.
2. (Classification of Equations)

The differential equation \( y' = f(x, y) \) is defined to be separable provided functions \( F \) and \( G \) exist such that \( f(x, y) = F(x)G(y) \).

(a) [40%] Check (x) the problems that can be converted into separable form. No details expected.

<table>
<thead>
<tr>
<th>( y' + 2xy = 3y )</th>
<th>( y' = 3xy^2 + (-xy + x^2y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' = \tan(x) + y )</td>
<td>( e^{2y}y' = xe^{2y} + (xe^y)^2 )</td>
</tr>
</tbody>
</table>

(b) [10%] Give an example \( y' = f(x, y) \) which is not separable, not quadrature, but it is linear.

(c) [20%] Apply a classification test to show that \( y' + \tan(x)y = -1 - x + xy \) is a linear differential equation. Supply all details, including a statement of the test.

(d) [30%] Apply a classification test to show that \( y' = y \sec(x + y) \) is not separable. Supply all details, including a statement of the test.

\[ \begin{align*}
\text{Eq. 1 (a)} & \quad y' = 3y - 2xy, \quad y' = (3-2x)y \quad \text{Sep} \\
\text{Eq. 2 (a)} & \quad y' = 3xy^2 - xy^2 + x^2y^2 = (3x - x + x^2)y^2 \quad \text{Sep} \\
\text{Eq. 3 (a)} & \quad y' = \tan(x) + y \quad \frac{dy}{dx} = \frac{1}{\tan x + y} \quad \text{at } y = 0 \text{ equals } \tan(x) \quad \text{Depends on } x \quad \text{Not Sep} \\
\text{Eq. 4 (a)} & \quad e^x e^y y' = xe^{2y} + x^2e^{2y} \quad \text{or } y' = (\frac{x^2+x^2}{e^x}) e^y \quad \text{Sep} \\
\end{align*} \]

(b) \( y' = \tan(x) + y \) \( \frac{dy}{dx} = 1 \neq 0 \Rightarrow \text{Not Quadrature, but Linear} \)

(c) \( \text{Test: } y' = f(x, y) \text{ is linear provided } \frac{\partial f}{\partial y} \text{ is independent of } y \)

Apply: \( f = -\tan(x)y - 1 - x + xy \) \( \frac{\partial f}{\partial y} = -\tan(x) + x \)

Independence of \( y \Rightarrow \text{Linear} \).

(d) \( \text{Test: } y' = f(x, y) \text{ not separable provided } \frac{\partial f}{\partial y} \text{ depends on } x \)

Apply: \( f = y \sec(x+y) \) \( \frac{\partial f}{\partial y} = \sec(x+y) + y \sec(x+y) \tan(x+y) \)

\( \frac{\partial f}{\partial y} = \frac{y}{1 + y \tan(x+y)} \quad \text{at } y = 1, \text{ get } \frac{1}{1 + \tan(x+1)} \text{ depends on } x \)

Use this page to start your solution. Attach extra pages as needed.

Then \( y' = y \sec(x+y) \) is not separable.
3. (Solve a Separable Equation)

Given \((xy + 2x + y + 2)y' = \left((1 + x)\sin^2(x)\cos(x) + 2x^2\right)(y^2 + 4y + 4)(y + 3)\).

(a) [80%] Find a non-constant solution in implicit form.  
To save time, do not solve for \(y\) explicitly. No answer check is expected.

(b) [20%] Find all constant solutions, which have the form \(y = c\), also called equilibrium solutions.  
No answer check is expected.

\[
\begin{align*}
(x+1)(y+2) y' &= \left((1+x)\sin^2 x \cos x + 2x^2\right)(y^2+4y+4)(y+3) \\
y' &= \left(\sin^2 x \cos x + \frac{2x^2}{1+x}\right)(y+2)(y+3) \\
\frac{y'}{(y+2)(y+3)} &= \sin^2 x \cos x + \frac{2x^2}{1+x} \\
\int (\text{RHS}) \, dx &= \sin^3 x + \int \frac{2(u-1)^2}{u} \, du \quad \text{when } u = 1+x \\
&= \sin^3 x + \int \left(\frac{2u^2-4u+2}{u}\right) \, du \\
&= \sin^3 x + u^2 - 4u + 2 \ln(u) + c_1 \\
&= \sin^3 x + (1+x)^2 - 4(x+1) + 2 \ln |x+1| + c_1 \\
&= \sin^3 x + (1+x)^2 - 4(x+1) + 2 \ln |x+1| + c_1 & \text{we'll absorb -4 into } c_1 \\
\int (\text{LHS}) \, dx &= \int \frac{du}{(u+2)(u+3)} \quad \text{when } u = y(x), \, du = y'(x) \, dx \\
&= \int \left(\frac{1}{u+2} + \frac{-1}{u+3}\right) \, du \\
&= \ln |y+2| - \ln |y+3| + c_2 \\
\end{align*}
\]

Answer (a): \( \ln |y+2| - \ln |y+3| = \frac{\sin^3 x}{3} + \frac{x^2 + 2x - 4x + (1-4)}{2\ln |1+x|} + c \) 
\( \text{can absorb into } c \)

Use this page to start your solution. Attach extra pages as needed.

Answer (b): \( y = -2 \) and \( y = -3 \)

To find \( \text{Roots, set } y' = 0 \) in (a) and solve for \( y \).
4. (Linear Equations)

(a) [50%] Solve the linear model $2x'(t) = -g + \frac{12}{2t+1} x(t)$, $x(0) = 0.125g$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $3x^2 \frac{dy}{dx} - 2y = 0$.

(c) [30%] Solve $30 \frac{dy}{dx} = 3 - 5y$ using the superposition principle $y = y_h + y_p$. Expected are answers for $y_h$ and $y_p$.

(a) 
\[
\begin{align*}
\chi' &= -\frac{6}{2t+1} \chi = -\frac{g}{2} \\
\frac{(WX)'}{W} &= -\frac{g}{2} \quad \text{when } W = e^{\int \frac{-6}{2t+1} dt} = e^{\ln |2t+1|^3} \\
(WX)' &= -\frac{g}{2} W \\
WX &= -\frac{g}{2} \int (2t+1)^{-3} dt \\
WX &= \frac{g}{8} (2t+1)^{-2} + C_1 \\
x &= \frac{g}{8} (2t+1)^{-2} + C_1 (2t+1)^3 \\
x(0) &= 0.125g = \frac{g}{8} \\
\text{also } x(0) &= \frac{g}{8} + C_1 \\
\text{so } C_1 &= 0 \quad \text{and } x(t) = \frac{g}{8} (2t+1)^3
\end{align*}
\]

(b) 
\[
\begin{align*}
y &= \frac{\text{constant}}{\text{integrating factor}} = \frac{c}{e^{\int \frac{-2}{3x^2} dx}} = \frac{c}{e^{\frac{2}{3x}}}
\end{align*}
\]

(c) 
\[
\begin{align*}
y_p &= \frac{3}{5} \\
y_h &= \frac{c}{e^{\int \frac{3}{30} dx}} = \frac{c}{e^{-x/6}} = c e^{-x/6} \\
y &= y_p + y_h = \frac{3}{5} + c e^{-x/6}
\end{align*}
\]

Use this page to start your solution. Attach extra pages as needed.
5. (Stability)  
(a) [50%] Draw a phase line diagram for the differential equation

\[
\frac{dx}{dt} = e^{x} (3 - |2x - 1|)(-2 + x)(x^2 - 4)(x - x^2)^2.
\]

Expected in the phase line diagram are equilibrium points and signs of \(dx/dt\).

\[
\begin{array}{cccccc}
+ & - & + & + & + & - \\
-2 & -1 & 0 & 1 & 2 & \\
\end{array}
\]

\(3 - |2x-1| = 0\)

At \(x = -1, 2\)

\(f(x) = e^{x} (3 - |2x-1|)(x-2)^2(x+2) x^2(l-x)^2\)

\(g(x) = (3 - |2x-1|)(x+2)\) determines the sign. Test \(-3, -1.5, -0.5, 0.5, 1.5, 3\)

(b) [50%] Assume an autonomous equation \(x'(t) = f(x(t))\). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

Use this page to start your solution. Attach extra pages as needed.