Differential Equations and Linear Algebra

2250-1 7:30am on 29 April 2014

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 1-2, 3, 4, 5, 6, 7, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter division adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronics are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader. More credit is possible if you write something.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted on **CANVAS**, found from the Registrar's web site link. Recording errors can be reported by email, hopefully as soon as discovered, but also even weeks after grades are posted.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Homework, quizzes and labs together count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

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Scores
Ch1-2.
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch9.
Ch10.

Ch1 and Ch2. (First Order Differential Equations) Complete all problems.

[20%] **Ch1-Ch2(a)**:

Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\Big((1+t^2)v(t)\Big) = t$, $v(0) = 0$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 100$.

Answer: $v(t) = \frac{1}{2}\frac{t^2}{1+t^2}$, $x(t) = \frac{1}{2}t - \frac{1}{2}\arctan(t) + 100$

[20%] **Ch1-Ch2(b)**:

Apply a test to show that $y' = e^x + y\sin(x)$ is not separable. Supply all details.

Answer: Use the partial derivative test for $y' = f(x,y)$, which says the equation is not separable provided f_y/f depends on x. In this case, $f(x,y) = e^x + y\sin(x)$, $f_y = \partial f/\partial y = \sin(x)$, $f_y/f = \sin(x)/(e^x + y\sin(x))$. When $y = 0$, then $f_y/f = e^{-x}\sin(x)$ depends on x, therefore the equation $y' = f(x,y)$ is not separable.

[20%] **Ch1-Ch2(c)**:

Solve the homogeneous equation $\dfrac{dy}{dx} - \dfrac{x}{1+x}y = 0$.

Answer: A constant divided by the integrating factor $W = e^{\int -\frac{x}{1+x}dx} = e^{\int -\frac{u-1}{u}du} = e^{-u+\ln|u|} = e^{-x+\ln|1+x|}$, where $u = 1+x$.

[20%] **Ch1-Ch2(d)**:

Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = (4-x)(x^2-16)(4-x^2)^3.$$

Label the equilibrium points, display the signs of dx/dt, and classify each equilibrium point as funnel, spout or node.

Answer: Equilibria $x = -4, -2, 2, 4$. Signs of dx/dt from left to right are **plus, plus, minus, plus, minus**. Spout at -4, Funnel at -2, Spout at 2, Node at 4.

[10%] **Ch1-Ch2(e)**:

The logistic model $\frac{dx}{dt} = x^2 - 6x + 5$ has two equilibrium solutions. Find them and identify which is the carrying capacity.

Answer: $x = 1, x = 5$; $x = 1$ is the carrying capacity.

[10%] **Ch1-Ch2(f)**:

Solve the linear drag model $10\dfrac{dv}{dt} = 40 - v$.

Answer: Superposition applies: $v = v_h + v_p$ where v_h solves $10v'(t) + v(t) = 0$ and v_p is the equilibrium solution. Then v_h = constant divided by the integrating factor $W = e^{t/10}$ and $v_p = 40$. Answer $v = ce^{-t/10} + 40$.

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Ch3. (Linear Systems and Matrices) Complete all problems.

[40%] **Ch3(a)**: Incorrect answers lose all credit. **Circle the correct answer**.

Ch3(a) Part 1. [10%]: True or False:
Assume given a 3×3 matrix A and 3×3 elementary matrices E_1, E_2, E_3. If $U = E_3E_2E_1A$ has nonzero determinant, then the system $A\vec{x} = \vec{b}$ has a unique solution for every 3×1 vector $\vec{b}$.

Ch3(a) Part 2. [10%]: True or False:
There is an example of a 3×3 matrix A such that the system $A\vec{x} = \vec{0}$ has solutions
$\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

Ch3(a) Part 3. [10%]: True or False:
If $n \times n$ matrix B was obtained from $n \times n$ matrix A by elementary row operations (`combo, swap, mult`), then B is the product of elementary matrices times A.

Ch3(a) Part 4. [10%]: True or False:
Given a 3×3 matrix A and a 3×1 vector $\vec{b}$, then the system $A\vec{x} = \vec{b}$ can be solved for $\vec{x}$ exactly when $\vec{b}$ is a linear combination of the pivot columns of A.

Answer: True. True. True. True.

[20%] **Ch3(b)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} 3 & 12 \\ -12 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

Answer: $x_1 = \Delta_1/\Delta$, $x_2 = \Delta_2/\Delta$, $\Delta = \det\begin{pmatrix} 3 & 12 \\ -12 & 3 \end{pmatrix} = 144 + 9 = 153$, $\Delta_1 = \det\begin{pmatrix} -1 & 12 \\ 2 & 3 \end{pmatrix} = -27$, $\Delta_2 = \det\begin{pmatrix} 3 & -1 \\ -12 & 2 \end{pmatrix} = -6$, $x_1 = \frac{-27}{153}$, $x_2 = \frac{-6}{153}$.

[40%] **Ch3(c)**: Determine which values of k correspond to: (1) A unique solution, (2) No solution, (3) Infinitely many solutions.
for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 8 & k & 0 \\ 4-k & 4-k & 0 \\ 8 & 10 & 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ -1 \\ k \end{pmatrix}.$$

Answer: (1) There is a unique solution for $\det(A) = 2(k-4)(8-k) \neq 0$, which implies $k \neq 4$ and $k \neq 8$. Therefore, the answer for infinitely many solutions, if there is one, must have either $k = 4$ or $k = 8$ or both. The no solution case must have a signal equation. The infinite solution case must have a free variable but no signal equation. Substitute $k = 4$ first to obtain augmented matrix

$$\left(\begin{array}{ccc|c} 8 & 4 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 8 & 10 & 2 & 4 \end{array}\right)$$

Substitute $k = 8$ second and then find the reduced row-echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -11/4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then (2) No solution for $k = 4$ [equation 2 is a signal equation]; (3) Infinitely many solutions for $k = 8$ [a free variable x_3 but no signal equation].

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Check the independence tests which apply to prove independence of three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in the vector space $\mathcal{R}^5$.

☐	**Wronskian test**	Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct Euler solution atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Sampling test**	Let samples a, b, c be given and for functions f, g, h define $$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$ Then $\det(A) \neq 0$ implies independence of f, g, h.

Answer: The tests that apply are the Rank test and the Pivot test. The Determinant test does not apply, because the augmented matrix of the three vectors is not square.

[20%] **Ch4(b)**: Consider the homogenous system $A\vec{x} = \vec{0}$. The nullity of A equals the number of free variables and the rank of A equals the number of lead variables. Mark the following statements as either TRUE or FALSE.

☐ The rank equals the number of pivots columns of A.

☐ The nullity equals the number of non-pivot columns of A.

☐ If the nullity is zero, then A is invertible.

☐ If the rank is zero, then A is the zero matrix.

Answer: True. True. False. True.

[40%] **Ch4(c)**: Let V be the vector space of all continuously differentiable vector functions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Let S be the set of all vector solutions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the dynamical system

$$\begin{cases} x'(t) &= 2x(t) + 3y(t), \\ y'(t) &= y(t) \end{cases}$$

Find two independent solutions $\vec{v}_1, \vec{v}_2$ such that $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$. *This calculation proves that S is a subspace of V by Picard's theorem and the Span Theorem, hence S is a vector space.*

Answer: The dynamical system is triangular, therefore it can be solved by the method of linear cascades (Section 1.5). This method solves first for $y = c_1 e^t$, then the answer is substituted into the first differential equation: $x' = 2x + 3c_1 e^t$. The integrating factor method applies to solve for $x = -3c_1 e^t + c_2 e^{2t}$. Two independent vector solutions are found by taking partial derivatives on the symbols c_1, c_2, obtaining $\vec{v}_1 = \begin{pmatrix} -3e^t \\ e^t \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$. Then any solution of the dynamical system is given by $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$.

A second method is Cayley-Hamilton-Ziebur. The eigenvalues of the matrix are $1, 2$, with corresponding Euler solution atoms e^t, e^{2t}. Then $x = c_1e^t + c_2e^{2t}$. Substitute x into $x' = 2x + 3y$ to find $y = -\frac{1}{3}c_1e^t$. Taking partials on symbols c_1, c_2 gives the vector solutions $\vec{v}_1 = \begin{pmatrix} e^t \\ -\frac{1}{3}e^t \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$. Then any solution of the dynamical system is given by $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$.

[20%] **Ch4(d)**: The 4×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Answer: Find $\mathbf{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 2 and 4.

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation roots of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = e^x\left(5x^2 + 20\sin(2x)\right)$ is a solution.

Answer: The atoms x^2e^x, $e^x\sin(2x)$ correspond to roots $0, 0, 0, 1 + 2i, 1 - 2i$. Although not requested, the root-factor theorem of college algebra implies the characteristic polynomial should be $r^3((r - 1)^2 + 4)$.

[20%] **Ch5(b)**: Determine a *basis of solutions* of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$(r^4 + 2r^2)(r^2 + 2r + 5) = 0.$$

Answer: The roots are $0, 0, \pm\sqrt{2}i, -1 \pm 2i$. By Euler's theorem, a basis is the set of atoms for these roots: $1, x, \cos(\sqrt{2}x), \sin(\sqrt{2}x), e^x\cos(2x), e^x\sin(2x)$.

[30%] **Ch5(c)**: Find the **steady-state periodic** solution for the forced damped spring-mass system

$$x'' + 2x' + 10x = 40\cos(2t).$$

Answer: It is known that the homogeneous solution has limit zero at $t = \infty$. Superposition $x = x_h + x + p$ then implies that the undetermined coefficients solution x_p is the steady-state periodic solution, with trial solution $x(t) = A\cos(2t) + B\sin(2t)$. Solving for undetermined coefficients A, B gives initially $-4x + 2x' + 10x = 40\cos(2t)$, then expanding $x' = -2A\sin(2t) + 2B\cos(2t)$ implies $6A\cos(2t) + 6B\sin(2t) + 2(-2A\sin(2t) + 2B\cos(2t)) = 40\cos(2t)$. Matching coefficients left and right gives the equations $6A + 4B = 40, -4A + 6B = 0$. Solving with Cramer's rule, $A = 60/13$, $B = 40/13$. The steady-state periodic solution is $x(t) = \frac{60}{13}\cos(2t) + \frac{40}{13}\sin(2t)$.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^5y}{dx^5} + 4\frac{d^3y}{dx^3} = 5x^2 + x + 7\sin 2x + 8xe^x$$

Answer: The homogeneous problem has roots $0, 0, 0, 2i, -2i$ with Euler solution atoms 1, x, x^2, $\cos(2x)$, $\sin(2x)$. The trial solution is constructed initially from $f(x) = 5x^2 + x + 7\sin 2x + 8xe^x$, which has a list of seven atoms in four groups (1) $1, x, x^2$; (2) $\cos 2x$; (3) $\sin 2x$; (4) e^x, xe^x. Conflicts with the homogeneous equation atoms causes a repair of groups (1), (2), (3), making the new groups (1) x^3, x^4, x^5; (2) $x\cos 2x$; (3) $x\sin 2x$; (4) e^{2x}, xe^{2x}. Then the shortest trial solution is a linear combination of the seven atoms in the corrected list.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[30%] **Ch6(a)**: Let A be a 2×2 matrix. Assume the system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$ has general solution

$$\vec{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Symbols c_1, c_2 are arbitrary constants. Find the matrix A.

Answer: Write $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Then $AP = PD$ implies $A = PDP^{-1} = \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix}$.

[30%] **Ch6(b)**: Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -8 & 1 & 1 \\ 1 & -8 & 1 \\ 0 & 0 & -8 \end{pmatrix}$$

Answer: Expand $|A - rI|$ by cofactors along row 3 to obtain a factored polynomial. The roots are $-9, -8, -7$.

[40%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 5 \end{pmatrix}$ is **not diagonalizable**. Display the details for computing all eigenpairs. *Details count 75%.*

Answer: The eigenpairs are $\left(0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right)$. Details below.

Expand $|A - rI| = r(r-3)^2$ by cofactor expansion on row 1. Then the eigenvalues are $0, 3, 3$. Label $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 3$.

The first eigenpair $(\lambda_1, \vec{v}_1)$ is found by solving $A\vec{v} = 0\vec{v}$, or equivalently, the homogeneous system $A\vec{v} = \vec{0}$. Then $\mathbf{rref}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} t_1 \\ 0 \\ 0 \end{pmatrix}$. The partial on symbol t_1 gives eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $\lambda_1 = 0$.

The second eigenpair $(\lambda_2, \vec{v}_2)$ is found by solving $A\vec{v} = 3\vec{v}$, or equivalently, the homogeneous system $(A - 3I)\vec{v} = \vec{0}$. Let $B = A - 3I = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{pmatrix}$. Then $\mathbf{rref}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and the solution is $\vec{v} = \begin{pmatrix} 0 \\ -2t_1 \\ t_1 \end{pmatrix}$. Take the partial on symbol t_1 to obtain the single answer $\vec{v}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

Eigenvalue $\lambda_3 = 3$ repeats the work for $\lambda_2 = 3$, in short, we have already computed all the answers for eigenvalue 3. The matrix has fewer than three eigenpairs: A is not diagonalizable.

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[20%] **Ch7(a)**: Incorrect answers lose all credit. **Circle the correct answer**.
Ch7(a) Part 1. [5%]: True or False:
A linear dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$ can always be solved by finding the eigenpairs of A.
Ch7(a) Part 2. [5%]: True or False:
A linear 2×2 dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$ has solution $\vec{u}(t)$ on $-\infty < t < \infty$ uniquely determined by the initial state $\vec{u}(0)$. The Cayley-Hamilton-Ziebur method finds the solution.
Ch7(a) Part 3. [5%]: True or False:
Linear systems $\frac{d}{dt}\vec{u} = A\vec{u}$ can always be solved by the linear integrating factor method, provided A is a triangular matrix.
Ch7(a) Part 4. [5%]: True or False:
Eigenpairs can be used to solve any second order system $\frac{d^2}{dt^2}\vec{u}(t) = A\vec{u}(t)$.

Answer: False. True. True. False.

[40%] **Ch7(b)**: Solve the 3×3 linear dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$, given matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use the most efficient method possible.

Answer: The matrix is triangular, so the theory of linear cascades applies. First, $x_3' = x_3$, then $x_3 = c_1 e^t$. Back-substitute: $x_2' = x_2 + 2x_3 = x_2 + 2c_1 e^t$ implies $(xW)'/W = 2c_1 e^t$ where $W = e^{-t}$ is the integrating factor. Then $x_2 = 2c_1 te^t + c_2 e^t$. Back-substitute into $x_1' = 2x_2$ to obtain $x_1' = 4c_1 te^t + 2c_2 e^t$,. Then integrate to obtain $x_1 = 4c_1(t-1)e^t + 2c_2 t + c_3$. The final answer is

$$\begin{aligned} x_1 &= 4c_1(t-1)e^t + 2c_2 t + c_3, \\ x_2 &= 2c_1 te^t + c_2 e^t, \\ x_3 &= c_1 e^t. \end{aligned}$$

[40%] **Ch7(c)**: Apply the eigenanalysis method to solve the system $\frac{d}{dt}\vec{u} = A\vec{u}$, when the matrix is defined as

$$A = \begin{pmatrix} -9 & 1 & 1 \\ 1 & -9 & 1 \\ 0 & 0 & -9 \end{pmatrix}$$

Answer: The eigenpairs of A are $(-10, \vec{v}_1), (-9, \vec{v}_2), (-8, \vec{v}_3)$ where

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\vec{u}(t) = c_1 e^{-10t}\vec{v}_1 + c_1 e^{-9t}\vec{v}_2 + c_1 e^{-8t}\vec{v}_3.$$

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{u}$$

Answer: It is a stable spiral. Details: The eigenvalues of A are roots of $r^2+2r+5 = (r+1)^2+4 = 0$, which are complex conjugate roots $-1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^{-t}\cos 2t$, $e^{-t}\sin 2t$ have limit zero at $t = \infty$, therefore the system is stable at $t = \infty$ and unstable at $t = -\infty$. So it must be a spiral [centers have no exponentials]. Report: **stable spiral**.

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -x - 2y^2 - y + 32, \\ y' &= 2x^2 + 2xy. \end{aligned}$$

An equilibrium point is $x = -4$, $y = 4$. Compute the Jacobian matrix $A = J(-4, 4)$ of the linearized system at this equilibrium point.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} -1 & -4y-1 \\ 4x+2y & 2x \end{pmatrix}$. Then $A = J(-4,4) = \begin{pmatrix} -1 & -17 \\ -8 & -8 \end{pmatrix}$.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -4x - 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{aligned} \tag{1}$$

At equilibrium point $x = 3$, $y = -3$, the Jacobian matrix is $A = J(3, -3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$.

[20%] Linear Problem: Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$.

[20%] Nonlinear Problem: Apply a theorem to classify $x = 3$, $y = -3$ as a saddle, center, spiral or node for the **nonlinear dynamical system** (1). Discuss all details of the application of the theorem. *Details count 75%.*

Answer: (1) The Jacobian is $J(x,y) = \begin{pmatrix} -4-2x & -4 \\ 3 & 3 \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$. The eigenvalues of A are found from $r^2 + 7r - 18 = 0$, giving distinct real roots $2, -9$. Because there are no trig functions in the Euler solution atoms e^{2t}, e^{-9t}, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms do not limit to zero at $t = \infty$ or $t = -\infty$, therefore it is a saddle and we report a **unstable saddle** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$. (2) Theorem 2 in Edwards-Penney section 9.2 applies to say that the same is true for the nonlinear system: **unstable saddle** at $x = 3$, $y = 3-$.

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Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\dfrac{-5}{s^3}$	$\dfrac{3}{4s+1}$	$\dfrac{-2s}{s^2+5}$	$\dfrac{1}{s^2+2s+5}$	$\dfrac{e^{-2s}}{s^2}$

$f(t)$	t^5	$\frac{1}{2}(1+e^{2t})e^{-t}$	$e^t\cos(2t)$	$e^{-t}\sin(3t)$	$te^t\cos t$
$\mathcal{L}(f(t))$					

Answer:
First table left to right:

$$\frac{-5t^2}{2}, \quad \tfrac{3}{4}e^{-t/4}, \quad -2\cos(\sqrt{5}t), \quad \tfrac{1}{2}e^{-t}\sin(2t), \quad (t-2)u(t-2).$$

The unit step $u(t)$ is defined in the Edwards-Penney textbook, $u(t)=1$ for $t \ge 0$, zero elsewhere.

Second table left to right:

$$\frac{120}{s^6}, \quad \frac{1}{2}\left(\frac{1}{s+1}+\frac{1}{s-1}\right), \quad \left.\frac{s}{s^2+4}\right|_{s\to(s-1)} = \frac{s-1}{(s-1)^2+4},$$
$$\left.\frac{3}{s^2+9}\right|_{s\to(s+1)} = \frac{3}{(s+1)^2+9}, \quad -\frac{d}{ds}\mathcal{L}(e^t\cos t) = -\frac{d}{ds}\frac{s-1}{(s-1)^2+1}$$

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for $f(t)=\sin(t)$ on $\pi \le t < 2\pi$ and $f(t)=0$ elsewhere.

Answer: Define $u(t)$ to be the unit step. Use $f(t) = \sin(t)(u(t-\pi) - u(t-2\pi))$ and the second shifting theorem. Then $\mathcal{L}(\sin(t)u(t-\pi)) = e^{-\pi s}\mathcal{L}(\sin(t)|_{t\to t+\pi}) = e^{-\pi s}\mathcal{L}(\sin(t+\pi)) = e^{-\pi s}\mathcal{L}(\sin(t)\cos(\pi)+\cos(t)\sin(\pi)) = -e^{-\pi s}\mathcal{L}(\sin(t)) = -e^{-\pi s}/(s^2+1)$. Similarly, $\mathcal{L}(\sin(t)u(t-2\pi)) = e^{-2\pi s}\mathcal{L}(\sin(t+2\pi)) = e^{-2\pi s}\mathcal{L}(\sin(t)) = e^{-2\pi s}/(s^2+1)$. The answer: $\mathcal{L}(f(t)) = -\left(e^{-\pi s}+e^{-2\pi s}\right)/(s^2+1)$.

[30%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{e^{-5s}}{(s-5)^2}$.

Answer: First write $\frac{1}{(s-5)^2} = \mathcal{L}(te^{5t})$ using the backward Laplace table and the first shifting theorem. We use the second shifting theorem: $e^{-as}\mathcal{L}(h(t)) = \mathcal{L}(h(t-a)u(t-a))$. Then $\mathcal{L}(f(t)) = e^{-5s}\mathcal{L}(te^{5t}) = \mathcal{L}(te^{5t}u(t)|_{t\to t-5}) = \mathcal{L}((t-5)e^{5t-25}u(t-5))$. Lerch's theorem implies $f(t) = (t-5)e^{5t-25}u(t-5)$.