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Differential Equations and Linear Algebra 2250

Midterm Exam 3

April 18, 2014

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 5)** Complete all.

(1a) [40%] The differential equation $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 6x$ has a particular solution $y_p(x)$ of the form $y = d_1x^2 + d_2x^3$. Find $y_p(x)$ by the method of undetermined coefficients (yes, find d_1, d_2).

Answer:

Solution y_h is a linear combination of the atoms $1, x, e^{-x}$. A particular solution is $y_p = x^3 - 3x^2$. The atoms for $y''' + y'' = 0$ are found from $r^3 + r^2 = 0$ with roots $r = 0, 0, -1$. The atoms in $f(x) = 6x$ are $1, x$. Because $1, x$ are solutions of the homogeneous equation, then the list $1, x$ from $f(x)$ is multiplied by x^2 to obtain the corrected list x^2, x^3. Then $y_p = d_1x^2 + d_2x^3$. Substitute y_p into the equation $y''' + y'' = 6x$ to get $6d_2x + 2d_1 + 6d_2 = 6x$. Matching coefficients of atoms gives $6d_2 = 6, 2d_1 + 6d_2 = 0$. Then $d_2 = 1, d_1 = -3$. Finally, $y_p = -3x^2 + x^3$.

(1b) [20%] Given $5x''(t) + \beta x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = \beta > 0$, $k = 4$, determine β such that the system is under-damped.
To save time, do not solve for $x(t)$!

Answer:

Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = \beta^2 - 4(5)(4) = \beta^2 - 80$. When there are two complex conjugate roots, then the equation is **under-damped**. This happens only in case the discriminant is negative, or $0 < \beta < \sqrt{80}$.

(1c) [40%] Given the forced spring-mass system $x'' + 4x' + 17x = 257\sin(4t)$, find the steady-state periodic solution.

Answer:

The answer is the undetermined coefficients solution $x_p(t) = A\cos(4t) + B\sin(4t)$, because the homogeneous solution $x_h(t)$ has limit zero at $t = \infty$. Substitute the trial solution into the differential equation. Then $A\cos(4t) + B\sin(4t) - 16A\sin(4t) + 16B\cos(4t) = 257\sin(4t)$. Matching coefficients of sine and cosine gives the equations $A + 16B = 0$, $-16A + B = 257$. Solving, $A = -16, B = 1$. Then $x_p(t) = -16\cos(4t) + \sin(4t)$ is the unique periodic steady-state solution.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Chapter 5)** Complete all.

(2a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of degree 5 with roots 0, $-\sqrt{2}$, $-\sqrt{2}$, $1+3i$, $1-3i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 5e^{-\sqrt{2}x} + 4x^2 + e^x \sin 3x$. Determine the undetermined coefficients **shortest** trial solution for y_p.
To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The Euler solution atoms for roots of the characteristic equation are $1, e^{-\sqrt{5}x}, xe^{-\sqrt{5}x}, e^x \cos 3x, e^x \sin 3x,$. The atom list for $f(x)$ is $e^{-\sqrt{5}x}$, $1, x, x^2$, $e^x \sin 3x$, $e^x \sin 3x$. This list of 6 atoms is broken into 4 groups, each group having exactly one base atom: (1) $1, x, x^2$, (2) $e^{-\sqrt{5}x}$, (3) $e^x \cos 3x$, (4) $e^x \sin 3x$. Each group contains a solution of the homogeneous equation. The modification rule is applied to groups 1 through 4. The trial solution is a linear combination of the replacement 6 atoms in the new list (1*) x, x^2, x^3, (2*) $x^2 e^{-\sqrt{5}x}$, (3*) $xe^x \cos 3x$, (4*) $xe^x \sin 3x$.

(2b) [40%] Let $f_1(x) = x^3 e^{-x}$, $f_2(x) = x^2 e^{-x} \sin(x)$. Mark with $\boxed{\textbf{YES}}$ or $\boxed{\textbf{NO}}$ the characteristic equations for which at least of f_1 or f_2 is an Euler solution atom.

☐ $r^2(r+1)^3(r-1)^3 = 0$

☐ $r(r^2-1)^4 = 0$

☐ $(r^2+1)^3(r+1)^4 = 0$

☐ $((r+1)^2+1)^2(r+1)^3 = 0$

Answer:

NO, YES, YES, NO. Function $f_1(x) = x^3 e^{-x}$ is an Euler solution atom for the differential equation if and only if $-1, -1, -1, -1$ are roots of the characteristic equation. Function $f_2(x) = x^2 e^{-x} \sin(x)$ is an Euler solution atom for the differential equation if and only if $-1 \pm i, -1 \pm i, -1 \pm i$ are roots of the characteristic equation.

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3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [40%] Display the details of Laplace's method to solve the system for $x(t)$ and $y(t)$.

$$\begin{aligned} x' &= -2x, \\ y' &= x + y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Graded details: (1) Forward transform; (2) Solve the system for $\mathcal{L}(x), \mathcal{L}(y)$; (3) Backward transform; (4) Answer for both $x(t), y(t)$.

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s+2)\mathcal{L}(x) + (0)\mathcal{L}(y) = 1, \quad (-1)\mathcal{L}(x) + (s-1)\mathcal{L}(y) = 2.$$

Any other method of arriving at the system of linear algebraic equations is acceptable, it is not an error to do it another way. Elimination or Cramer's rule or matrix inversion applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{s-1}{(s-1)(s+2)} = \dfrac{1}{s+2}$. Then the backward table implies $x(t) = e^{-2t}$. Similarly, $\mathcal{L}(y(t)) = \dfrac{2s+5}{(s-1)(s+2)} = \dfrac{-1/3}{s+2} + \dfrac{7/3}{s-1}$. Then the backward table implies $y(t) = -\frac{1}{3}e^{-2t} + \frac{7}{3}e^{t}$.

(3b) [20%] Find $f(t)$,

$$\mathcal{L}(f(t)) = \frac{5s^2 - s - 1}{s^2(s-1)}.$$

Graded details: (1) Partial fractions; (2) Backward table

Answer:

The numerator has degree 3, less than the denominator degree 4. Partial fraction theory gives an expansion $\frac{5s^2-s-1}{s^2(s-1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-1}$. We find the system of equations and solve it for $a = 2$, $b = 1$, $c = 3$. Then $\mathcal{L}(f(t)) = \frac{2}{s} + \frac{1}{s^2} + \frac{3}{s-1} = \mathcal{L}(2 + t + 3e^t)$ implies $f(t) = 2 + t + 3e^t$.

(3c) [40%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(e^t f(t)) = \frac{2}{(s+1)^3}.$$

Graded details: (1) Laplace theorem(s); (2) Shift theorem; (3) Forward table; (4) Backward table; (5) Answer.

Answer:

(1) Use the s-differentiation theorem, $\mathcal{L}((-t)^2 e^t f(t)) = \frac{2}{(s+1)^3}$; (2) Shift theorem, $\frac{2}{(s+1)^3} = \left.\frac{2}{s^3}\right|_{s=s+1}$; (3) Backward table, $\frac{2}{(s+1)^3} = \left.\mathcal{L}(t^2)\right|_{s=s+1}$; (4) Shift theorem, $\frac{2}{(s+1)^3} = \mathcal{L}(t^2 e^{-t})$; (5) Lerch, $t^2 e^t f(t) = t^2 e^{-t}$, or $f(t) = e^{-2t}$.

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4. **(Chapter 10)** Complete all parts.

(4a) [60%] Fill in the blank spaces in the Laplace table. Partial credit is given for wrong answers, based on documented steps.

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^t\sin(\pi t)$	
$(2t+1)^2$	
$t^2e^{\pi t}$	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{3}{s^2+9}$	$\sin 3t$
$\dfrac{1}{s^2+2s+5}$	
$\dfrac{1}{(3s+1)^2}$	
$\dfrac{s+1}{(s-1)^2}$	

Answer:

Forward: $\dfrac{\pi}{(s-1)^2+\pi^2}$, $\mathcal{L}(4t^2+4t+1) = \dfrac{8}{s^3}+\dfrac{4}{s^2}+\dfrac{1}{s}$, $\frac{2}{(s-\pi)^3}$.

Backward: $\frac{1}{2}e^{-t}\sin(2t)$, $(1/9)te^{-t/3}$, $e^t + 2te^t$.

(4b) [40%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 4x(t) = 5e^t, \quad x(0) = x'(0) = 0.$$

Graded details: (1) Transform; (2) Isolate $\mathcal{L}(x(t))$; (3) Partial fractions; (4) Backward table; (5) Answer.

Answer:

(1) $(s^2+4)\mathcal{L}(x) = 5/(s-1)$; (2) $\mathcal{L}(x) = \frac{5}{(s-1)(s^2+4)}$; (3) $\mathcal{L}(x) = \frac{1}{s-1} - \frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4}$; (4) $\mathcal{L}(x) = \mathcal{L}(e^t - \cos(2t) - \frac{1}{2}\sin(2t))$; (5) $x(t) = e^t - \cos(2t) - \frac{1}{2}\sin(2t)$.

Use this page to start your solution. Attach extra pages as needed.

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5. (Chapter 6) Complete all parts.

(5a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 5 & 11 & 1 & -6 \\ -3 & -7 & 2 & -7 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer:

The eigenvalues are $-1, -5, 2 \pm 3i$.
Subtract λ from the diagonal. Expand the determinant by the cofactor rule along column 4. Collect terms on the common factor $(2-\lambda)^2 + 9$.

(5b) [40%] Given matrix $A = \begin{pmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{pmatrix}$ and one eigenpair $\left(3, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right)$, find the other eigenpairs.
Graded details: (1) Eigenvalues; (2) Eigenvector details; (3) Answer.

Answer:

(1) Eigenvalues $-2, -2, 3$. (2) One frame sequence is required for $\lambda = -2$. The sequence starts with $\begin{pmatrix} 0 & 0 & 0 \\ -5 & 0 & -5 \\ 5 & 0 & 5 \end{pmatrix}$, the last frame having two rows of zeros. There are two invented symbols t_1, t_2 in the last frame algorithm answer $x_1 = -t_2$, $x_2 = t_1$, $x_3 = t_2$. (3) Taking ∂_{t_1} and ∂_{t_2} gives two eigenvectors, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(5c) [30%] Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 3×3 linear differential system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{0t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where c_1, c_2, c_3 are arbitrary constants (e^{0t} has exponent zero). Find a matrix multiply formula for the matrix A. Then check the answer

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Answer:

The eigenvalues come from the exponentials, $2, 2, 0$. The eigenpairs are $\left(2, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(0, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Test $AP = PD$ against the answer $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed.