Differential Equations and Linear Algebra 2250  
Midterm Exam 2

**Instructions:** This in-class exam is designed to be completed in 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
   Let \( a, b \) and \( c \) denote constants and consider the system of equations
   \[
   \begin{pmatrix}
   0 & 0 & 0 \\
   -2b - 4 & 3 & a \\
   b + 1 & -1 & 0 \\
   -1 - b & 1 & a
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y \\
   z
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 \\
   b^2 \\
   b \\
   b^2 - b
   \end{pmatrix}
   \]

   (a) \([40\%]\) Determine \( a \) and \( b \) such that the system has a unique solution.

   (b) \([30\%]\) Explain why \( a = 0 \) and \( b \neq 0 \) implies no solution. Ignore any other possible no solution cases.

   (c) \([30\%]\) Explain why \( a = b = 0 \) implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

   Because of a row of zeros (equation 0 = 0), we can reduce the question to a 3x3 problem \( \vec{A} \vec{u} = \vec{b} \) when

   \[
   \vec{A} = \begin{pmatrix}
   -2b - 4 & 3 & a \\
   b + 1 & -1 & 0 \\
   -1 - b & 1 & a
   \end{pmatrix}, \quad \vec{b} = \begin{pmatrix}
   b^2 \\
   b \\
   b^2 - b
   \end{pmatrix}, \quad \vec{u} = \begin{pmatrix}
   x \\
   y \\
   z
   \end{pmatrix}
   \]

   \[\text{Rank } |\vec{A}| = a - ab = a(1-b) \neq 0 \text{ for } a \neq 0 \text{ and } b \neq 1\]

   (a) Unique sol when \( |\vec{A}| \neq 0 \) (see reduction above)

   (b) when \( a = 0 \), then \( \vec{A} = \begin{pmatrix}
   -2b - 4 & 3 & 0 \\
   b + 1 & -1 & 0 \\
   -1 - b & 1 & 0
   \end{pmatrix} \) has col rank \( \leq 2 \)

   hence there is one free var, if the system is consistent. We do

   REF steps on \( C = \langle A | \vec{b} \rangle \) to get

   \[
   \begin{pmatrix}
   2b - 4 & 3 & 0 & b^2 \\
   b + 1 & -1 & 0 & b \\
   0 & 0 & 0 & b^2 - b
   \end{pmatrix}
   \]

   hence no solution for \( b \neq 0 \),

   (c) Define \( C \) as in part (b), then substitute \( a = b = 0 \) to get

   REF step \( \begin{pmatrix}
   -4 & 3 & 0 & 0 \\
   1 & -1 & 0 & 0 \\
   0 & 0 & 0 & 0
   \end{pmatrix} \). A homogeneous system is consistent.

   Use this page to start your solution. Attach extra pages as needed.
2. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is a subspace $S$ of $\mathbb{R}^3$ containing none of the vectors $(1, 1, 3)$ and $(1, -1, 1)$. 

(b) [10%] True or false: The set of solutions $\vec{u}$ in $\mathbb{R}^3$ of a consistent matrix equation $A\vec{u} = \vec{b}$ can equal all vectors in the $xy$-plane, that is, all vectors of the form $\vec{u} = (x, y, 0)$. 

(c) [10%] True or false: Relations $x^2 + y^2 = 0$, $y + z = 0$ define a subspace in $\mathbb{R}^3$.

(d) [10%] True or false: Equations $z = y$, $z = 2y$ define a subspace in $\mathbb{R}^3$.

(e) [20%] Linear algebra theorems are able to conclude that the set $S$ of all polynomials $f(x) = c_0 + c_1x + c_2x^2$ such that $f'(x) + f(1)f(x)dx = 0$ is a vector space of functions. Explain why $V = \text{span}(1, x, x^2)$ is a vector space, then fully state a linear algebra theorem required to show $S$ is a subspace of $V$. To save time, do not write any subspace proof details.

(f) [40%] Find a basis of vectors for the subspace of $\mathbb{R}^5$ given by the system of restriction equations

$$
\begin{align*}
3x_1 & + 2x_3 + 4x_4 + 10x_5 = 0, \\
2x_1 & + x_3 + 2x_4 + 4x_5 = 0, \\
-2x_1 & + x_3 + 4x_5 = 0, \\
2x_1 & + 2x_3 + 4x_4 + 12x_5 = 0.
\end{align*}
$$

(c) $x^2 + y^2 = 0 \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$ which are 2 linear algebraic eqs. Rest follows from the Kernel Theorem.

(e) $V = \text{span}(1, x, x^2) = \text{subspace of } \mathbb{R}^3 \text{ vector space of all polynomials, by the Span Theorem (Sec. 4.2)}$

$S$ is a subspace by the Subspace Criterion (Section 4.1) Theorem $S$ is a subspace of vector space $V$ provided

(1) $\vec{0}$ is in $S$; (2) $\vec{x}, \vec{y}$ in $S \Rightarrow \vec{x} + \vec{y}$ in $S$; (3) $\vec{x}$ in $S$, $c = \text{const} \Rightarrow c\vec{x}$ in $S$

(f) Variable $x_2$ missing, so it's a free variable. Augmented matrix is

$$
C = \begin{pmatrix}
3 & 0 & 2 & 4 & 10 & 0 \\
2 & 0 & 1 & 2 & 4 & 0 \\
-2 & 2 & 0 & 0 & 4 & 0 \\
2 & 0 & 2 & 12 & 4 & 0
\end{pmatrix}, \quad \text{ref}(C) = \begin{pmatrix}
1 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

Use this page to start your solution. Attach extra pages as needed.
3. (Independence and Dependence) Do all parts.
   (a) [10%] State a dependence test for 3 vectors in \( \mathbb{R}^4 \). Write the hypothesis and conclusion, not just the name of the test.
   (b) [10%] State fully an independence test for 3 polynomials. It should apply to show that \( 1, 1+x, x(x+1) \) are independent.
   (c) [10%] For any matrix \( A \), \( \text{rank}(A) \) equals the number of lead variables for the problem \( A\vec{x} = \vec{0} \). How many non-pivot columns in an \( 8 \times 8 \) matrix \( A \) with \( \text{rank}(A) = 6 \)?
   (d) [30%] Let \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) denote the rows of the matrix
   \[
   A = \begin{pmatrix}
   0 & -2 & 0 & -6 & 0 \\
   0 & 2 & 0 & 5 & 1 \\
   0 & 1 & 0 & 2 & 1 \\
   0 & 1 & 0 & 3 & 0 
   \end{pmatrix}
   \]
   Decide if the four rows \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) are independent and display the details of the chosen independence test.
   (e) [40%] Extract from the list below a largest set of independent vectors.
   \[
   \begin{align*}
   \vec{v}_1 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix},
   \vec{v}_2 &= \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 0 \end{pmatrix},
   \vec{v}_3 &= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix},
   \vec{v}_4 &= \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \\ 3 \end{pmatrix},
   \vec{v}_5 &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},
   \vec{v}_6 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}.
   \end{align*}
   \]
   \[\text{(a) Rank Test : 3 vectors } \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are dependent } \iff \text{The rank of the augmented matrix has rank } < 3.\]
   \[\text{(b) Wronskian Test : If the Wronskian of the 3 polynomials is nonzero at some } x = x_0, \text{ then the 3 polynomials are independent.}\]
   \[\text{(c) The Rank = # pivot cols, Therefore } \text{Ans = 2}\]
   \[\text{(d) The col rank is } \leq 3, \text{ so the 4 rows cannot be independent, by the Rank Test applied to the transpose of } A.\]
   \[\text{(e) The safest method is to form the augmented matrix } A \text{ of the col vectors (} A \text{ is } 6 \times 6). \text{ Then find the pivot cols from } \text{ref}(A). \text{ The answer is } \begin{pmatrix} \vec{v}_2, \vec{v}_4 \end{pmatrix}. \text{ It is also possible to see } \vec{v}_3 = \frac{1}{2} \vec{v}_2, \vec{v}_5 = \vec{v}_4 - \vec{v}_2, \vec{v}_6 = \vec{v}_5 - \vec{v}_3.
   \]
   \[\text{So } \vec{v}_2, \vec{v}_4 \text{ are the independent columns.}\]

Use this page to start your solution. Attach extra pages as needed.
Instructions: This in-class exam is designed to be completed in under 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Determinants) Do all parts.
   (a) [10%] True or False. The value of a determinant is the product of the diagonal elements.
   (b) [10%] True or False. The determinant of the negative of the $n \times n$ identity matrix is $-1$.
   (c) [20%] Assume given $3 \times 3$ matrices $A$, $B$. Suppose $A^2B = E_2 E_1 A$ and $E_1$, $E_2$ are elementary matrices representing respectively a swap and a multiply by $-5$. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.
   (d) [30%] Determine all values of $x$ for which $(I + C)^{-1}$ fails to exist, where $C$ equals the transpose of the matrix \[
\begin{pmatrix}
2 & 0 & -1 \\
3x & 0 & 1 \\
x-1 & x & x
\end{pmatrix}.
\]
   (e) [30%] Let symbols $a, b, c$ denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of $A^{-1}$, given $A$ below.

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
a & b & 0 & 1 \\
1 & c & 1 & 2
\end{pmatrix}
\]

\[\det(A) = 1 E_2 E_3 A \Rightarrow |A|^2 = |B| = |E_2| |E_3| |A| \text{ by the determinant product theorem. Then } |B| = 10, |E_2| = -5, |E_3| = -1 \text{ implies that } |A|^2 (10) = (-5)(-1)|A| \Rightarrow \frac{-5(2|A|^2-|A|)}{5} = 10 \Rightarrow A = 0 \text{ or } x = \frac{1}{2} \text{. Then } |A| = |2A| = |2I| |A| = 8 |A| = \begin{pmatrix} 0 & 4 \end{pmatrix}.
\]

4. (Determinants) Do all parts.
   (a) [10%] True or False. The value of a determinant is the product of the diagonal elements.
   (b) [10%] True or False. The determinant of the negative of the $n \times n$ identity matrix is $-1$.
   (c) [20%] Assume given $3 \times 3$ matrices $A$, $B$. Suppose $A^2B = E_2 E_1 A$ and $E_1$, $E_2$ are elementary matrices representing respectively a swap and a multiply by $-5$. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.
   (d) [30%] Determine all values of $x$ for which $(I + C)^{-1}$ fails to exist, where $C$ equals the transpose of the matrix \[
\begin{pmatrix}
2 & 0 & -1 \\
3x & 0 & 1 \\
x-1 & x & x
\end{pmatrix}.
\]
   (e) [30%] Let symbols $a, b, c$ denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of $A^{-1}$, given $A$ below.

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
a & b & 0 & 1 \\
1 & c & 1 & 2
\end{pmatrix}
\]

Consider \[\det(A) = 1 E_2 E_3 A \Rightarrow |A|^2 = |B| = |E_2| |E_3| |A| \text{ by the determinant product theorem. Then } |B| = 10, |E_2| = -5, |E_3| = -1 \text{ implies that } |A|^2 (10) = (-5)(-1)|A| \Rightarrow \frac{-5(2|A|^2-|A|)}{5} = 10 \Rightarrow A = 0 \text{ or } x = \frac{1}{2} \text{. Then } |A| = |2A| = |2I| |A| = 8 |A| = \begin{pmatrix} 0 & 4 \end{pmatrix}.
\]

(b) \[|I + C^T| = |I^T + C^T| = |(I + C)^T| = |I + C| = \begin{pmatrix} 3 & 0 & -1 \\
3x & 1 & 1 \\
x-1 & x+1 & x-3x^2+2
\end{pmatrix} = \begin{pmatrix} x = 1, x = -2/3 \end{pmatrix}.
\]

(c) \[\text{entry } = \text{ cofactor}(A_{1,4,3})/|A| = (-1)^{3+4}|\text{minor}(A_{1,4,3})/|A| = \begin{pmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
a & b & 1
\end{pmatrix} = \begin{pmatrix} -1 \end{pmatrix} \text{. Then } |A| = \begin{pmatrix} 1 & 0 & 0 \\
1 & 0 & 0 \\
a & b & 1
\end{pmatrix} = \begin{pmatrix} -1 \end{pmatrix} = 1 \text{ (c) of (b) at } \begin{pmatrix} 3 \end{pmatrix}.
\]

Use this page to start your solution. Attach extra pages as needed.
5. (Linear Differential Equations) Do all parts.
(a) [20%] Solve for the general solution of \(15y'' + 8y' + y = 0\).
(b) [40%] The characteristic equation is \(r^2(2r+1)^3(r^2 - 2r + 10) = 0\). Find the general solution \(y\) of the linear homogeneous constant-coefficient differential equation.
(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions \(2e^{3x} + 4x\) and \(xe^{3x}\). Write a formula for the general solution.
(d) [20%] Mark with \(\mathbf{X}\) the functions which cannot be a solution of a linear homogeneous differential equation with constant coefficients. Test your choices against this theorem:

The general solution of a linear homogeneous nth order differential equation with constant coefficients is a linear combination of Euler solution atoms.

| \(X\) | \(X = e^{ln(2x)}\) | \(X = e^{x^2}\) | \(2\pi + x\) | \(\cos(ln|x|)\) |
|-------|------------------|------------------|-------------|-------------|
| \(\cos(x \ln(3.7125))\) | \(x^{-1}e^{-x} \sin(\pi x)\) | \(\cosh(x)\) | \(\sin^2(x)\) |

(a) \(15r^2 + 8r + 1 = 0 \Rightarrow (5r + 1)(3r + 1) = 0 \Rightarrow r = -\frac{1}{5}, -\frac{1}{3}\)
\(y = c_1 \ e^{-x/5} + c_2 \ e^{-x/3}\)

(b) roots: \(r = 0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 1 \pm 3i\)
Euler sol atoms = \(1, x, e^{\frac{x}{2}}, x \ e^{-\frac{x}{2}}, x^2 \ e^{-x/2}, e^x \ cos(3x), e^x \ sin(3x)\)
\(y = \text{linear combination of the atoms}\)

(c) Atoms must be \(1, x, e^{3x}, xe^{3x}\) and the general solution is a linear combination of the Euler sol. atoms.

(d) \(e^{ln(2x)} = 12x\) not an atom or l.c. of atoms
\(cos(ln|x|)\) not an atom, only \(cos(bx)\) qualifies
\(x^{-1}\) not allowed, only positive powers or zero power
\(e^{x^2}\) not of the form \(e^{ax}\).

Use this page to start your solution. Attach extra pages as needed.