Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 2

Scores

Instructions: This in-class exam is designed to be completed in 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
   Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 0 & 0 & 0 \\ -2b-4 & 3 & a \\ b+1 & -1 & 0 \\ -1-b & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ b^2 \\ b \\ b^2-b \end{pmatrix}$$

   (a) [40%] Determine a and b such that the system has a unique solution.

   (b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.

   (c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

Because of a row of zeros (equation $0=0$), we can reduce the question to a 3×3 problem $A\vec{u} = \vec{B}$ where

$$A = \begin{pmatrix} -2b-4 & 3 & a \\ b+1 & -1 & 0 \\ -1-b & 1 & a \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} b^2 \\ b \\ b^2-b \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then $|A| = a - ab = a(1-b) \neq 0$ for $a \neq 0$ and $b \neq 1$

(a) unique sol when $|A| \neq 0$ (see reduction above) $\boxed{a \neq 0 \text{ and } b \neq 1}$

(b) when $a=0$, then $A = \begin{pmatrix} -2b-4 & 3 & 0 \\ b+1 & -1 & 0 \\ -1-b & 1 & 0 \end{pmatrix}$ has col rank ≤ 2

hence there is one free var, if the system is consistent. We do rref steps on $C = \langle A | B \rangle$ to get $\left(\begin{array}{ccc|c} 2b-4 & 3 & 0 & b^2 \\ b+1 & -1 & 0 & b \\ 0 & 0 & 0 & b^2 \end{array}\right)$

There is a <u>signal eq</u> for $b \neq 0$, hence no solution for $a=0, b\neq 0$.

(c) Define C as in part (b), then substitute $a=b=0$ to get rref step $\left(\begin{array}{ccc|c} -4 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. A homogeneous system is consistent. One free variable implies ∞-many solutions.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

(a) [10%] $\boxed{\text{True}}$ or false: There is a subspace S of $\mathcal{R}^3$ containing none of the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}$.

$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2 \Rightarrow \dim(S) = 1$, $S = \mathrm{span}(\vec{v}_1 \times \vec{v}_2)$

(b) [10%] $\boxed{\text{True}}$ or false: The set of solutions $\vec{u}$ in $\mathcal{R}^3$ of a consistent matrix equation $A\vec{u} = \vec{b}$ can equal all vectors in the xy-plane, that is, all vectors of the form $\vec{u} = (x, y, 0)$. Example: $\begin{cases} z = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$

(c) [10%] $\boxed{\text{True}}$ or false: Relations $x^2 + y^2 = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

(d) [10%] $\boxed{\text{True}}$ or false: Equations $x = y$, $z = 2y$ define a subspace in $\mathcal{R}^3$. Kernel Theorem

(e) [20%] Linear algebra theorems are able to conclude that the set S of all polynomials $f(x) = c_0 + c_1 x + c_2 x^2$ such that $f'(x) + \int_0^1 f(x)x\,dx = 0$ is a vector space of functions. Explain why $V = \mathrm{span}(1, x, x^2)$ is a vector space, then fully state a linear algebra theorem required to show S is a subspace of V. To save time, do not write any subspace proof details.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^5$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl}
3x_1 & + & 2x_3 & + & 4x_4 & + & 10x_5 & = & 0, \\
2x_1 & + & x_3 & + & 2x_4 & + & 4x_5 & = & 0, \\
-2x_1 & & & & & + & 4x_5 & = & 0, \\
2x_1 & + & 2x_3 & + & 4x_4 & + & 12x_5 & = & 0.
\end{array}$$

(c) $x^2 + y^2 = 0 \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$ which are 2 linear algebraic eqs.
Rest follows from the Kernel Theorem.

(e) $V = \mathrm{span}(1, x, x^2)$ = subspace of the vector space of all polynomials, by the SPAN Theorem (Sec. 4.2)

S is a subspace by the Subspace Criterion (Section 4.1)

Theorem S is a subspace of vector space V provided
(1) $\vec{0}$ is in S; (2) $\vec{x}, \vec{y}$ in $S \Rightarrow \vec{x} + \vec{y}$ in S; (3) $\vec{x}$ in S, c = const $\Rightarrow c\vec{x}$ in S

(f) variable x_2 missing, so it's a free variable. Augmented matrix is

$$C = \left(\begin{array}{ccccc|c} 3 & 0 & 2 & 4 & 10 & 0 \\ 2 & 0 & 1 & 2 & 4 & 0 \\ -2 & 0 & 0 & 0 & 4 & 0 \\ 2 & 0 & 2 & 4 & 12 & 0 \end{array}\right), \quad \mathrm{rref}(C) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Last frame alg scalar answer is
$$\begin{cases} x_1 = 2t_3 \\ x_2 = t_1 \\ x_3 = -2t_2 - 8t_3 \\ x_4 = t_2 \\ x_5 = t_3 \end{cases}$$

Use this page to start your solution. Attach extra pages as needed.

$$\text{Basis} = \{\partial_{t_1}, \partial_{t_2}, \partial_{t_3}\} = \left\{ \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-8\\0\\1\end{pmatrix} \right\}$$

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3. **(Independence and Dependence)** Do **all parts.**

(a) [10%] State a dependence test for 3 vectors in $\mathcal{R}^4$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State fully an independence test for 3 polynomials. It should apply to show that 1, $1+x$, $x(1+x)$ are independent.

(c) [10%] For any matrix A, **rank**(A) equals the number of lead variables for the problem $A\vec{x} = \vec{0}$. How many non-pivot columns in an 8×8 matrix A with **rank**$(A) = 6$?

(d) [30%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & -2 & 0 & -6 & 0 \\ 0 & 2 & 0 & 5 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{pmatrix}.$$

Decide if the four rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are independent and display the details of the chosen independence test.

(e) [40%] Extract from the list below a largest set of independent vectors.

$$\vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \vec{v_5} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \vec{v_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

(a) Rank Test: 3 vectors v_1, v_2, v_3 are dependent $\iff$ The rank of the augmented matrix has rank < 3.

(b) Wronskian Test: If the Wronskian of the 3 polynomials is nonzero at some $x = x_0$, then the 3 polynomials are independent.

(c) The rank = # pivot cols, therefore $\boxed{\text{ans} = 2}$

(d) The col rank is ≤ 3, so the 4 rows cannot be independent, by the rank test applied to the transpose of A.

(e) The safest method is to form the augmented matrix A of the col vectors (A is 6×6). Then find the pivot cols from rref(A). The answer is $\boxed{\vec{V_2}, \vec{V_4}}$. It is also possible to see $V_3 = \frac{1}{2} V_2$, $V_5 = V_4 - V_2$, $V_6 = V_5 - V_3$. So V_2, V_4 are the independent columns.

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4. **(Determinants)** Do all parts.

(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.

(b) [10%] True or False? The determinant of the negative of the $n \times n$ identity matrix is -1.

(c) [20%] Assume given 3×3 matrices A, B. Suppose $A^2B = E_2E_1A$ and E_1, E_2 are elementary matrices representing respectively a swap and a multiply by -5. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.

(d) [30%] Determine all values of x for which $(I + C)^{-1}$ fails to exist, where C equals the transpose of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 3x & 0 & 1 \\ x-1 & x & x \end{pmatrix}$.

(e) [30%] Let symbols a, b, c denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

(c) $|A^2B| = |E_2E_1A| \Rightarrow |A|^2|B| = |E_2||E_1||A|$ by The det product Thm. Then $|B| = 10$, $|E_2| = -5$, $|E_1| = -1$ implies That $|A|^2(10) = (-5)(-1)|A|$ or $5(2|A|^2 - |A|) = 0$. So $|A| = 0$ or $|A| = 1/2$. Then $|C| = |2A| = |2I||A| = 8|A| = \boxed{0 \text{ or } 4}$

(d) $|I + C^T| = |I^T + C^T| = |(I+C)^T| = |I + C| = \begin{vmatrix} 3 & 0 & -1 \\ 3x & 1 & 1 \\ x-1 & x & x+1 \end{vmatrix} =$
$(3)\begin{vmatrix} 1 & 1 \\ x & x+1 \end{vmatrix} + (-1)\begin{vmatrix} 3x & 1 \\ x-1 & x \end{vmatrix} = 3 - (3x^2 - x + 1) = x - 3x^2 + 2 =$
$(-3x - 2)(x - 1)$. Ans: $\boxed{x = 1,\ x = -2/3}$

(e) entry $=$ cofactor$(A,4,3)/|A| = (-1)^{3+4}$ minor$(A,4,3)/|A| = \boxed{1}$

minor$(A,4,3) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ a & b & 1 \end{vmatrix} = -1$, $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ a & b & 1 \end{vmatrix}(-1) = 1$ (cof on col 3)

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5. (Linear Differential Equations) Do all parts.

(a) [20%] Solve for the general solution of $15y'' + 8y' + y = 0$.

(b) [40%] The characteristic equation is $r^2(2r+1)^3(r^2 - 2r + 10) = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.

(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions $2e^{3x} + 4x$ and xe^{3x}. Write a formula for the general solution.

(d) [20%] Mark with **X** the functions which **cannot** be a solution of a linear homogeneous differential equation with constant coefficients. Test your choices against this theorem:

The general solution of a linear homogeneous nth order differential equation with constant coefficients is a linear combination of Euler solution atoms.

X $e^{\ln\|2x\|}$	X e^{x^2}	$2\pi + x$	$\cos(\ln\|x\|)$ X
$\cos(x \ln\|3.7125\|)$	$x^{-1}e^{-x}\sin(\pi x)$ X	$\cosh(x)$	$\sin^2(x)$

(a) $15r^2 + 8r + 1 = 0 \Rightarrow (5r+1)(3r+1) = 0 \Rightarrow r = -1/5, -1/3$

$y = c_1 e^{-x/5} + c_2 e^{-x/3}$

(b) roots: $r = 0, 0, -1/2, -1/2, -1/2, 1 \pm 3i$

Euler Sol atoms $=$ $1, x, e^{-x/2}, xe^{-x/2}, x^2e^{-x/2}, e^x\cos 3x, e^x \sin 3x$

$y =$ linear combination of the atoms

(c) Atoms must be $1, x, e^{3x}, xe^{3x}$ and the general solution is a linear combination of the 4 Euler Sol. atoms.

(d) $e^{\ln|2x|} = |2x|$ not an atom or l.c. of atoms

$\cos(\ln|x|)$ not an atom, only $\cos(bx)$ qualifies

x^{-1} not allowed, only positive powers or zero power

e^{x^2} not of the form e^{ax}.