**Differential Equations and Linear Algebra 2250**

**Midterm Exam 2**

**Version 1, 21 Mar 2014**

**Instructions:** This in-class exam is designed to be completed in 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (The 3 Possibilities with Symbols)
   Let $a$, $b$ and $c$ denote constants and consider the system of equations
   
   
   \[
   \begin{pmatrix}
   b - 3 & a & 2 \\
   2b - 4 & a & 3 \\
   1 - b & 0 & -1 \\
   b - 1 & a & 1
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y \\
   z
   \end{pmatrix} =
   \begin{pmatrix}
   b^2 - b \\
   b^2 \\
   -b \\
   b^2 + b
   \end{pmatrix}
   \]
   
   Plan: Toolkit until a row of zeros. Then replace by a $3 \times 3$ system, take determinant.

   (a) [40%] Determine $a$ and $b$ such that the system has a unique solution.
   
   (b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.
   
   (c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

2. Consider $a = 0$ in $1\alpha$

   \[
   \begin{pmatrix}
   b - 3 & 0 & 0 \\
   0 & 0 & -1 \\
   0 & 0 & -b
   \end{pmatrix}
   \]

   Use this page to start your solution. Attach extra pages as needed.
2. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: The span of the vectors \((1, 1, 1), (2, 3, 1), (1, 0, 2)\) is a subspace of \(\mathbb{R}^3\) of dimension 2.  

(b) [10%] True or false: A matrix equation \(Ax = b\) has a solution \(x\) if and only if the matrix \(A\) has an inverse.

(c) [10%] True or false: Relation \(x + y = 2x + z\) defines a subspace in \(\mathbb{R}^3\).

(d) [10%] True or false: The span of \(e^x, e^{-x}, \sinh(x)\) is a subspace of the vector space of functions continuous on \(-\infty < x < \infty\).

(e) [20%] State two linear algebra theorems that apply to conclude that the set \(S = \text{span}(1, x, x^2)\) is a subspace of the vector space of all polynomials. Do not present details, but please state the theorems fully.

(f) [40%] Find a basis of vectors for the subspace of \(\mathbb{R}^5\) given by the system of restriction equations

\[
\begin{align*}
2x_1 + x_2 + 2x_4 + 4x_5 &= 0, \\
3x_1 + 2x_2 + 4x_4 + 10x_5 &= 0, \\
-x_1 + 4x_4 &= 0, \\
2x_1 + 2x_2 + 4x_4 + 12x_5 &= 0.
\end{align*}
\]

2(e) Span Theorem. The span of vectors \(\vec{v}_1, \ldots, \vec{v}_k\) in vector space \(V\) is a subspace \(S\) of \(V\).

Subspace Criterion. A subset \(S\) of \(V\) is a subspace of \(V\) provided:

1. \(\vec{0}\) is in \(S\); 
2. if \(\vec{v}_i, \vec{v}_j \in S\) then \(\vec{v}_i + \vec{v}_j \in S\); 
3. if \(c \in \mathbb{C}\) then \(c \vec{v} \in S\).

2(f) Solve the homogeneous system using the Last Frame Algorithm.

\[
\begin{bmatrix}
2 & 1 & 0 & 2 & 4 & 0 \\
3 & 0 & 0 & 10 & 4 & 0 \\
-2 & 0 & 0 & 4 & 12 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 1 & 0 & 2 & 4 & 0 \\
0 & 1 & 0 & 2 & 8 & 0 \\
0 & 0 & 0 & 8 & 0 & 0
\end{bmatrix} \text{ combo}(3, 4, -1) \rightarrow \begin{bmatrix}
2 & 0 & 0 & 2 & 4 & 0 \\
0 & 1 & 0 & 2 & 8 & 0 \\
0 & 0 & 0 & 8 & 0 & 0
\end{bmatrix} \text{ combo}(1, 2, -1) \rightarrow \begin{bmatrix}
1 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0
\end{bmatrix} \text{ combo}(3, 2, -1)
\]

Use this page to start your solution. Attach extra pages as needed.

\[
\text{Basis} = \left\{ \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} \right\} = \left\{ \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} \right), \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} \right), \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} \right) \right\}
\]
3. (Independence and Dependence) Do all parts.

(a) [10%] State an independence test for 3 vectors in $\mathbb{R}^5$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State fully an independence test for 3 functions, each is which is a linear combination of Euler solution atoms, e.g., $e^x, \cosh(x), \sin(x) + x$. No details are expected, but please state the test fully.

(c) [40%] Let $v_1, v_2, v_3, v_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & -2 & -6 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{pmatrix}.$$  

Decide if the four rows $v_1, v_2, v_3, v_4$ are independent and display the details of the chosen independence test.

(d) [40%] Extract from the list below a largest set of independent vectors.

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, v_5 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, v_6 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$  

3(a) The 3 vectors are independent $\iff$ The rank of their augmented matrix is 3.

3(b) If the Wronskian determinant of the 3 functions is nonzero for some $x$, then the 3 functions are independent.

3(c) $\det \begin{pmatrix} 0 & -2 & -6 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{pmatrix} = 0$ because of zero col 1. Determinant test implies the rows are dependent, because $|A| = |A^T|$.

3(d) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 2 \\ 0 & 2 & 1 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 1 & 3 & 0 \\ 0 & -2 & -1 & -1 & 2 \\ 0 & 2 & 1 & 3 & 5 \end{pmatrix}$ combo $(2,3,-1)$ and swaps

$\begin{pmatrix} 0 & 2 & -1 & -1 & 2 \\ 0 & 2 & 1 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Use this page to start your solution. Attach extra pages as needed.

$\{v_2, v_4\}$ = largest indep set.
4. (Determinants) Do all parts.
   (a) [10%] True or False? The toolkit operations of swap, combination and multiply apply to determinants and leave the value of the determinant unchanged.
   (b) [10%] True or False? For 3 x 3 matrices A and B, \(|A + B| = |A| + |B|\).
   (c) [20%] Assume given 3 x 3 matrices A, B. Suppose \(A^2B = E_3E_2E_1\) and \(E_1, E_2, E_3\) are elementary matrices representing respectively a combination, a multiply by \(-3\) and a swap. Assume \(\det(B) = 9\). Let \(C = 3A\). Find all possible values of \(\det(C)\).
   (d) [30%] Suppose matrix \(C\) is 8 x 8, nonzero but \(C^2 = 0\). Let \(B = I - C\) and \(A = I + C\). Explain why \(B\) is the inverse of \(A\).
   (e) [30%] Let symbols \(a, b, c\) denote constants. Define matrix

   \[
   A = \begin{pmatrix}
   1 & 1 & 0 & a \\
   1 & 0 & 0 & 0 \\
   a & b & 0 & 1 \\
   1 & c & 1 & 2
   \end{pmatrix}
   \]

   Assume \(|A| \neq 0\). Then the adjugate [adjoint] formula for the inverse is defined. Find the value of the entry in row 2, column 3 of \(A^{-1}\).

\[
4(\text{c}) \quad |A|^2 |B| = |E_3||E_2||E_1| \Rightarrow |A|^2 |a| = (-1)(-3)(1) \Rightarrow |A|^2 = \frac{3}{27} = \frac{1}{9}
\]

\[
|C| = |(3I)(A)| = 13 |I||A| = 27 |A| = \left\{ \begin{array}{l}
\frac{27}{\sqrt{3}} \\
-\frac{27}{\sqrt{3}}
\end{array} \right.
\]

\[
4(\text{b}) \quad AB = (I+C)(I-C) = I + C - C - C^2 = I - C^2 = I
\]

So \(A, B\) are inverses. We use Theorem That \(AB = I \Rightarrow BA = I\).

\[
4(\text{e}) \quad \text{entry in } A^{-1} \text{ in row 2, col 3} = \frac{\text{cofactor}(A, 3, 2)}{|A|}
\]

\[
\begin{vmatrix}
1 & 1 & 0 & a \\
1 & 0 & 0 & 0 \\
a & b & 0 & 1 \\
1 & c & 1 & 2
\end{vmatrix} = \frac{-a}{|1-a|}
\]

Use this page to start your solution. Attach extra pages as needed.
5. (Linear Differential Equations) Do all parts.
(a) [20%] Solve for the general solution of $12y'' + 11y' + 2y = 0$.
(b) [40%] The characteristic equation is $(4r^4 - r^2)(r^2 - 2r + 5)^2 = 0$. Find the general solution $y$ of the linear homogeneous constant-coefficient differential equation.
(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has a particular solution $2e^{-x} + \sin(x) + xe^{-x}$. Find the roots of the characteristic equation.
(d) [20%] Mark with $\times$ the functions which can possibly be a solution of a linear homogeneous differential equation with constant coefficients. Test your choices against this theorem:

*The general solution of a linear homogeneous nth order differential equation with constant coefficients is a linear combination of Euler solution atoms.*

| $\sin(|x|)$ | $\times$ | $2x\pi$ | $\cos(x\pi)$ |
|-------------|-----------|---------|---------------|
| $\cos(x/e^2)$ | $\times$ | $e^{-x}\sin(10\pi x)$ | $x\sin(x)$ |
| $x + e^{-2x^2}(x)$ | | | |

-2 for each error
Low score = 8

5(a) $12r^4 + 11r + 2 = (4r+1)(3r^2 + 2)$, roots $= -1/4, -2/3$

$y = c_1 e^{-x/4} + c_2 e^{-2x/3}$

5(b) $r^2(4r^2-1)((r-1/2)^2 + 1)^2 = r^2(2r-1)(2r+1)((r-1)/2 + 1)^2$

roots $= 0, 0, 1/2, -1/2, 1 \pm 2i, 1 \pm 2i$

atoms $= 1, x, e^{x/2}, e^{-x/2}, e^{x}\cos(2x), e^{x}\sin(2x), xe^{x/2}, xe^{-x/2}, xe^{x}\sin(2x)$

$y = \text{linear combination of the atoms above}$

5(c) roots $= -1, -1, i, -i$

5(d) Remarks

$\sin(1x1)$ not an atom
$x^{m}$ disallowed in an atom - only positive integer power

$e^{x}\sin(10\pi x)$ is an atom, but no fraction $\frac{1}{x}$ allowed.

Use this page to start your solution. Attach extra pages as needed.