Name KEY

Scores
1.
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Differential Equations and Linear Algebra 2250

Midterm Exam 2
Version 1, 21 Mar 2014

Instructions: This in-class exam is designed to be completed in 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} b-3 & a & 2 \\ 2b-4 & a & 3 \\ 1-b & 0 & -1 \\ b-1 & a & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b^2-b \\ b^2 \\ -b \\ b^2+b \end{pmatrix}$$

Plan: Toolkit until a row of zeros, Then replace by a 3x3 system, take determinant.

(a) [40%] Determine a and b such that the system has a unique solution.

(b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.

Plan: produce a signal equation.

(c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases. plan: Consistent system + one free variable

1(a) $\begin{pmatrix} b-3 & a & 2 & b^2-b \\ 2b-4 & a & 3 & b^2 \\ 1-b & 0 & -1 & -b \\ 0 & a & 0 & b^2 \end{pmatrix}$ combo(3,4,1) $\rightarrow \begin{pmatrix} b-3 & 0 & 2 & -b \\ 2b-4 & 0 & 3 & 0 \\ 1-b & 0 & -1 & -b \\ 0 & a & 0 & b^2 \end{pmatrix}$ combo(4,1,-1) combo(4,2,-1)

$\rightarrow \begin{pmatrix} b-3 & 0 & 2 & -b \\ b-3 & 0 & 2 & -b \\ 1-b & 0 & -1 & -b \\ 0 & a & 0 & b^2 \end{pmatrix}$ combo(3,2,1) $\rightarrow \begin{pmatrix} b-3 & 0 & 2 & -b \\ 1-b & 0 & -1 & -b \\ 0 & a & 0 & b^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo(1,2,-1) Swap zero row to end

Consider equivalent system $\left(\begin{array}{ccc|c} b-3 & 0 & 2 & -b \\ 1-b & 0 & -1 & -b \\ 0 & a & 0 & b^2 \end{array}\right)$ Det of Coeff $= \begin{vmatrix} b-3 & 0 & 2 \\ 1-b & 0 & -1 \\ 0 & a & 0 \end{vmatrix}$

Unique for $a(b+1) \neq 0$. Det $= -a(-b+3+2b-2)$

1(b) Consider $a=0$ in 1(a)

$\left(\begin{array}{ccc|c} b-3 & 0 & 2 & -b \\ 1-b & 0 & -1 & -b \\ 0 & 0 & 0 & -b^2 \end{array}\right)$

Signal equation from row 3, if $b \neq 0$. No Sol.

1(c) Consider $a=b=0$ in 1(a)

$\left(\begin{array}{ccc|c} -3 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ consistent. 2 lead vars. 1 free var.

∞ - Many solutions.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: The span of the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}$ is a subspace of $\mathcal{R}^3$ of dimension 2.

$\begin{vmatrix}1 & 1 & 3\\ 1 & -1 & 1\\ 1 & 0 & 2\end{vmatrix} = 0 \Rightarrow$ dependent, but first 2 are indep.

(b) [10%] True or false: A matrix equation $A\vec{x} = \vec{b}$ has a solution $\vec{x}$ if and only if the matrix A has an inverse.

A non-square, $\vec{b} = \vec{0}$, 1+ free variable

(c) [10%] True or false: Relation $x + y = 2x + z$ defines a subspace in $\mathcal{R}^3$. Kernel Thm

(d) [10%] True or false: The span of $e^x, e^{-x}, \sinh(x)$ is a subspace of the vector space of functions continuous on $-\infty < x < \infty$. span Thm

(e) [20%] State two linear algebra theorems that apply to conclude that the set $S = \mathbf{span}(1, x, x^2)$ is a subspace of the vector space of all polynomials. Do not present details, but please state the theorems fully.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^5$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl}
2x_1 & + & x_2 & + & 2x_4 & + & 4x_5 & = & 0,\\
3x_1 & + & 2x_2 & + & 4x_4 & + & 10x_5 & = & 0,\\
-2x_1 & & & & & + & 4x_5 & = & 0,\\
2x_1 & + & 2x_2 & + & 4x_4 & + & 12x_5 & = & 0.
\end{array}$$

2(e) Span Theorem. The span of vectors $\vec{v}_1, \ldots, \vec{v}_k$ in vector space V is a subspace S of V.

Subspace Criterion. A subset S of V is a subspace of V provided (1) $\vec{0}$ is in S; (2) $\vec{v}_1, \vec{v}_2$ in $S \Rightarrow \vec{v}_1 + \vec{v}_2$ in S; (3) c = constant and $\vec{v}$ in $S \Rightarrow c\vec{v}$ in S.

2(f) Solve the homogeneous system using the Last Frame Algorithm

$$\left(\begin{array}{ccccc|c} 2 & 1 & 0 & 2 & 4 & 0\\ 3 & 2 & 0 & 4 & 10 & 0\\ -2 & 0 & 0 & 0 & 4 & 0\\ 2 & 2 & 0 & 4 & 12 & 0\end{array}\right) \to \left(\begin{array}{ccccc|c} 2 & 1 & 0 & 2 & 4 & 0\\ 3 & 2 & 0 & 4 & 10 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\end{array}\right) \begin{array}{l}\text{combo}(1,3,1)\\ \text{combo}(1,4,-1)\end{array} \to$$

$$\left(\begin{array}{ccccc|c} 2 & 1 & 0 & 2 & 4 & 0\\ 3 & 2 & 0 & 4 & 10 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right) \text{combo}(3,4,-1) \to \left(\begin{array}{ccccc|c} 2 & 1 & 0 & 2 & 4 & 0\\ 1 & 1 & 0 & 2 & 6 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right) \text{combo}(1,2,-1) \to$$

$$\left(\begin{array}{ccccc|c} 2 & 0 & 0 & 0 & -4 & 0\\ 1 & 1 & 0 & 2 & 6 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right) \text{combo}(3,1,-1) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0\\ 1 & 0 & 0 & 0 & -2 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right) \begin{array}{l}\text{mult}(1, 1/2)\\ \text{combo}(3,2,-1)\end{array}$$

$$\text{RREF} = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0\\ 0 & 1 & 0 & 2 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right) \to \begin{cases} x_1 - 2x_5 = 0\\ x_2 + 2x_4 + 8x_5 = 0\\ 0 = 0\\ 0 = 0\end{cases} \begin{cases} x_1 = 2t_3\\ x_2 = -2t_2 - 8t_3\\ x_3 = t_1\\ x_4 = t_2\\ x_5 = t_3\end{cases}$$

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$$\text{Basis} = \left\{\frac{\partial\vec{x}}{\partial t_1}, \frac{\partial\vec{x}}{\partial t_2}, \frac{\partial\vec{x}}{\partial t_3}\right\} = \left\{\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-8\\0\\0\\1\end{pmatrix}\right\}$$

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3. **(Independence and Dependence)** Do **all parts.**

(a) [10%] State an independence test for 3 vectors in $\mathcal{R}^5$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State fully an independence test for 3 functions, each is which is a linear combination of Euler solution atoms, e.g., $e^x, \cosh(x), \sin(x) + x$. No details are expected, but please state the test fully.

(c) [40%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & -2 & -6 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{pmatrix}.$$

Decide if the four rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are independent and display the details of the chosen independence test.

(d) [40%] Extract from the list below a largest set of independent vectors.

$$\vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \vec{v_5} = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 5 \end{pmatrix}, \vec{v_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}.$$

3(a) The 3 vectors are independent $\Leftrightarrow$ The rank of their augmented matrix is 3.

3(b) If the Wronskian determinant of the 3 functions is nonzero for some x, then the 3 functions are independent.

3(c) $\begin{vmatrix} 0 & -2 & -6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{vmatrix} = 0$ because of zero col 1. Determinant test implies the rows are dependent, because $|A| = |A^T|$.

3(d) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & -2 & -1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 3 & 5 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & -2 & -1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 3 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ combo(2,3,-1) and swaps

$\to \begin{pmatrix} 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}$ combo(1,2,1) combo(1,3,-1) $\to \begin{pmatrix} 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

pivot = 2 pivot = 4

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$\{\vec{v}_2, \vec{v}_4\}$ = largest indep set.

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4. **(Determinants)** Do all parts.

(a) [10%] True or False? The toolkit operations of swap, combination and multiply apply to determinants and leave the value of the determinant unchanged. only combo unchanged

(b) [10%] True or False? For 3×3 matrices A and B, $|A+B| = |A| + |B|$. Ex : $A = I$, $B = -I$, 2X2

(c) [20%] Assume given 3×3 matrices A, B. Suppose $A^2B = E_3E_2E_1$ and E_1, E_2, E_3 are elementary matrices representing respectively a combination, a multiply by -3 and a swap. Assume $\det(B) = 9$. Let $C = 3A$. Find all possible values of $\det(C)$.

(d) [30%] Suppose matrix C is 8×8, nonzero but $C^2 = 0$. Let $B = I - C$ and $A = I + C$. Explain why B is the inverse of A.

(e) [30%] Let symbols a, b, c denote constants. Define matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

Assume $|A| \neq 0$. Then the adjugate [adjoint] formula for the inverse is defined. Find the value of the entry in row 2, column 3 of A^{-1}.

4(c) $|A|^2|B| = |E_3||E_2||E_1| \Rightarrow |A|^2(9) = (-1)(-3)(1) \Rightarrow |A|^2 = \frac{1}{3}$

$|C| = |(3I)(A)| = |3I||A| = 27|A| = \begin{cases} 27/\sqrt{3} \\ -27/\sqrt{3} \end{cases}$

4(d) $AB = (I+C)(I-C) = I + C - C - C^2 = I - C^2 = I$

So A, B are inverses. we use the Theorem That $AB = I \Rightarrow BA = I$.

4(e) entry in A^{-1} in row=2, col=3 $= \dfrac{\text{cofactor}(A,3,2)}{|A|}$

$$\text{entry} = \frac{\begin{vmatrix} 1 & 0 & a \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix}(-1)^{2+3}}{\begin{vmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{vmatrix}} = \frac{(-1)^3\begin{vmatrix} 1 & a \\ 1 & 0 \end{vmatrix}}{(-1)\begin{vmatrix} 1 & 0 & a \\ b & 0 & 1 \\ c & 1 & 2 \end{vmatrix}} = \frac{-a}{(-1)(-1)\begin{vmatrix} 1 & a \\ b & 1 \end{vmatrix}} = \frac{-a}{1-ab}$$

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5. (Linear Differential Equations) Do all parts.

(a) [20%] Solve for the general solution of $12y'' + 11y' + 2y = 0$.

(b) [40%] The characteristic equation is $(4r^4 - r^2)(r^2 - 2r + 5)^2 = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.

(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has a particular solution $2e^{-x} + \sin(x) + xe^{-x}$. Find the roots of the characteristic equation.

(d) [20%] Mark with **X** the functions which **can possibly be** be a solution of a linear homogeneous differential equation with constant coefficients. Test your choices against this theorem:

The general solution of a linear homogeneous nth order differential equation with constant coefficients is a linear combination of Euler solution atoms.

$\sin(\lvert x\rvert)$	$\frac{x}{e^{x}}$ X	$2x^{\pi}$	$\cos(x - \pi)$ X
$\cos(x/e^2)$ X	$\frac{e^{-x}\sin(10\pi x)}{x}$	$x\sinh(x)$ X	$x + \cos^2(x)$ X

−2 for each error
Low score = 8

5(a) $12r^2 + 11r + 2 = (4r+1)(3r+2)$, roots $= -1/4, -2/3$

$y = c_1 e^{-x/4} + c_2 e^{-2x/3}$

5(b) $r^2(4r^2-1)((r-1)^2+4)^2 = r^2(2r-1)(2r+1)((r-1)^2+4)^2$

roots $= 0, 0, \frac{1}{2}, -\frac{1}{2}, 1 \pm 2i, 1 \pm 2i$

atoms $= 1, x, e^{x/2}, e^{-x/2}, e^x\cos 2x, e^x\sin 2x, xe^x\cos 2x, xe^x\sin 2x$

y = linear combination of the atoms above

5(c) roots $= -1, -1, i, -i$

5(d) Remarks

$\sin(|x|)$ not an atom

x^{π} disallowed for an atom - only positive integer powers

$e^{-x}\sin(10\pi x)$ is an atom, but no fraction $\frac{1}{x}$ allowed.

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