Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 1

Version 1, 14 Feb 2013

Scores
1.
2.
3.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Problem 1(d): Typo 20*exp(t) changed to 20*exp(-t) in class on exam day.

1. **(Quadrature Equations)**

(a) [25%] Solve $y' = \dfrac{x^2-2}{1+x}$.

(b) [25%] Solve $y' = \dfrac{1}{(\cos x + 1)(\cos x - 1)}$.

(c) [25%] Solve $y' = \dfrac{(1+e^{2x})^2}{e^x}$, $y(0) = 1$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(e^{2t}v(t)\right) = 20e^t$, $v(0) = 0$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 100$.

(a) $1+x = u$, $\dfrac{x^2-2}{1+x} = \dfrac{(u-1)^2-2}{u} = \dfrac{u^2-2u-1}{u} = u - 2 - 1/u$

$y = \int y'dx = \int u\,dx = u^2/2 - 2u - \ln|u| + c = \boxed{\dfrac{(1+x)^2}{2} - 2x - \ln|1+x| + c_1}$,

Also by long division, $\dfrac{x^2-2}{x+1} = x - 1 - 1/(x+1) \Rightarrow \boxed{y = \dfrac{x^2}{2} - x - \ln|1+x| + c_2}$

(b) $(\cos x + 1)(\cos x - 1) = \cos^2 x - 1 = -\sin^2 x$

$y' = -\csc^2 x \Rightarrow y = \int y' = \int -\csc^2 x = \boxed{\cot x + c}$

(c) $y' = \frac{1}{e^x}(1 + 2e^{2x} + e^{4x}) = e^{-x} + 2e^x + e^{3x}$

$y = \int y' = \int (e^{-x} + 2e^x + e^{3x})\,dx = \boxed{-e^{-x} + 2e^x + \frac{1}{3}e^{3x} + c}$

(d) $e^{2t}v = \int \frac{d}{dt}(e^{2t}v)dt = \int 20e^{-t}dt = -20e^{-t} + c$

$v(0) = 0 \Rightarrow 0 = -20 + c \Rightarrow c = 20.$ $\boxed{v = -20e^{-3t} + 20e^{-2t}}$

$x = \int v = \int(-20e^{-3t} + 20e^{-2t})dt = \frac{20}{3}e^{3t} - 10e^{-2t} + c_1$

$x(0) = 100 \Rightarrow 100 = \frac{20}{3} - 10 + c_1 \Rightarrow c_1 = \frac{310}{3}$

$\boxed{x(t) = \frac{20}{3}e^{-3t} - 10e^{-2t} + \frac{310}{3}}$

Use this page to start your solution. Attach extra pages as needed.

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided functions F and G exist such that $f(x, y) = F(x)G(y)$.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

$\square$	$y' + xy = y^2 + xy^2$	$\boxed{X}$	$y' = (x-1)(y+1) + (y - xy)y$
$\square$	$y' = \cos(xy)$	$\boxed{X}$	$e^x y' = x \ln\lvert y\rvert + x^2 \ln(y^2)$

(b) [10%] State a partial derivative test that concludes $y' = f(x, y)$ is a linear differential equation but not a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = xy^2$ is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + y \ln\lvert x\rvert$ is not separable. Supply all details.

(a) $y' = -xy + y^2(1+x)$ not sep

$y' = xy - y + x - 1 + y^2 - xy^2$
$= (x-1)y + (1-x)y^2 + (x-1)$
$= (x-1)(y - y^2 + 1)$ separable

$y' = \cos(xy)$ not sep

$e^x y' = x \ln\lvert y\rvert + 2x^2 \ln\lvert y\rvert$
$= (x + 2x^2)\ln\lvert y\rvert$ separable

(b) $\frac{\partial f}{\partial y} \neq 0$ and independent of $y \Rightarrow$ linear and not quadrature

(c) $f(x,y) = xy^2 \Rightarrow \frac{\partial f}{\partial y} = 2xy$ not independent of $y \Rightarrow$ Not Linear

(d) $f(x,y) = e^x + y \ln\lvert x\rvert \Rightarrow \frac{f_y}{f} = \frac{\ln\lvert x\rvert}{e^x + y\ln\lvert x\rvert}$ depends on x,

because $\left.\frac{f_y}{f}\right|_{x=1} = 0$ and $\left.\frac{f_y}{f}\right|_{x=2} = \frac{\ln 2}{e^2 + y \ln 2} \neq 0$

Therefore, $y' = f(x,y)$ is not separable.

Use this page to start your solution. Attach extra pages as needed.

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3. (Solve a Separable Equation)
Given $(yx + 2y)y' = ((2 + x)\sin(x)\cos(x) + x)(y^2 + 3y + 2)$.

(a) [80%] Find a non-constant solution in implicit form.
To save time, **do not solve** for y explicitly. No answer check expected.

(b) [20%] Find all constant solutions (also called equilibrium solutions; no answer check expected).

(a) $(x+2)yy' = [(x+2)\sin x \cos x + x](y+2)(y+1)$

$$\frac{yy'}{(y+2)(y+1)} = \sin x \cos x + \frac{x}{x+2}$$

$$\int LHS = \int \left(\frac{A}{y+2} + \frac{B}{y+1}\right)y' = A\ln|y+2| + B\ln|y+1| + c_1$$

$$\frac{y}{(y+2)(y+1)} = \frac{A}{y+2} + \frac{B}{y+1} \Rightarrow A = 2,\ B = -1$$

$$\int RHS = \int \sin x \cos x\, dx + \int \left(1 - \frac{2}{x+2}\right)dx$$

$$= \frac{\sin^2 x}{2} + x - 2\ln|x+2| + C_2$$

Implicit sol

$$\boxed{2\ln|y+2| - \ln|y+1| = \frac{1}{2}\sin^2 x + x - 2\ln|x+2| + c}$$

(b) Find $y =$ constant in $y' = F(x)G(y)$ means $0 = F(x)G(y)$ or $G(y) = 0$. Here, $G(y) = \dfrac{(y+2)(y+1)}{y}$, so $\boxed{y = -1,\ y = -2}$

Use this page to start your solution. Attach extra pages as needed.

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Scores
4.
5.

Differential Equations and Linear Algebra 2250

Midterm Exam 1a

Version 1a, 21 Feb 2013

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Linear Equations)

(a) [50%] Solve the linear velocity model. Show all integrating factor steps.

$$\begin{cases} 5v'(t) &= -160 + \dfrac{12}{2t+1}v(t), \\ v(0) &= 80 \end{cases}$$

(b) [25%] Solve the homogeneous equation $\dfrac{dy}{dx} - \left(\dfrac{1}{x} + \cos(x)\right) y = 0$.

(c) [25%] Solve $4\dfrac{dy}{dx} - 24y = \dfrac{2}{\pi}$ using the superposition principle $y = y_h + y_p$. Expected are three answers, y_h, y_p and y.

(a) $5v' = -160 + \dfrac{12}{2t+1} v$

$v' = -32 + \dfrac{12}{10t+5} v$

(DE) $v' - \dfrac{12}{10t+5} v = -32$

$W = e^{\int p\,dt} = e^{-\frac{6}{5}\ln|10t+5|}$

$W = (10t+5)^{-6/5}$ (t near 0)

$\dfrac{(vW)'}{W} = -32$ Replace LHS of (DE)

$vW = -32\int W + c$ Quadr.

$= -32 \dfrac{(10t+5)^{-1/5}}{(-1/5)(10)} + c$

$v = 16(10t+5) + c(10t+5)^{6/5}$

$v(0) = 80 \Rightarrow c = 0$

$v = 16(10t+5) = \boxed{160t + 80}$

ans check: IC ✓ DE ✓

$160 \overset{?}{=} -32 + 12(16)$ yes.

(b) $y = \dfrac{c}{W}$ Shortcut

$W = e^{\int p\,dx} = e^{-\ln|x| - \sin x}$

$y = \dfrac{cx}{e^{-\sin x}} = \boxed{cxe^{\sin x}}$

(c) Equil. Sol.: $y_p = \dfrac{-2}{24\pi}$

Homog. Sol.: $y' - 6y = 0$

$y_h = \dfrac{c}{e^{-6x}}$

Sol: $y = y_h + y_p$

$= \boxed{ce^{6x} - \dfrac{1}{12\pi}}$

ans check:

$4y' - 24y = 24ce^{6x} - 24y$

$= -24\left(-\dfrac{1}{12\pi}\right)$

$= \dfrac{2}{\pi}$ ✓

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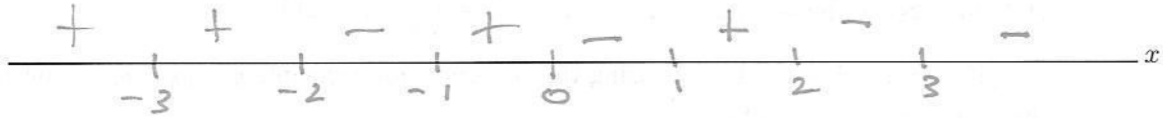
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

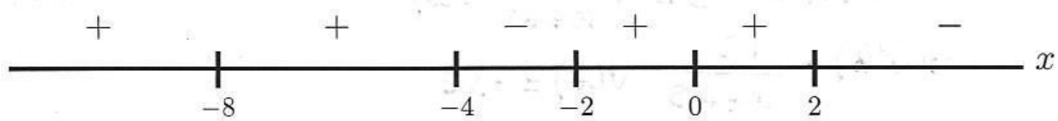
$$\frac{dx}{dt} = \sinh(x+1)\,(2-|4x-2|)^3\,(3-|x|)(x^2-9)(4-x^2).$$

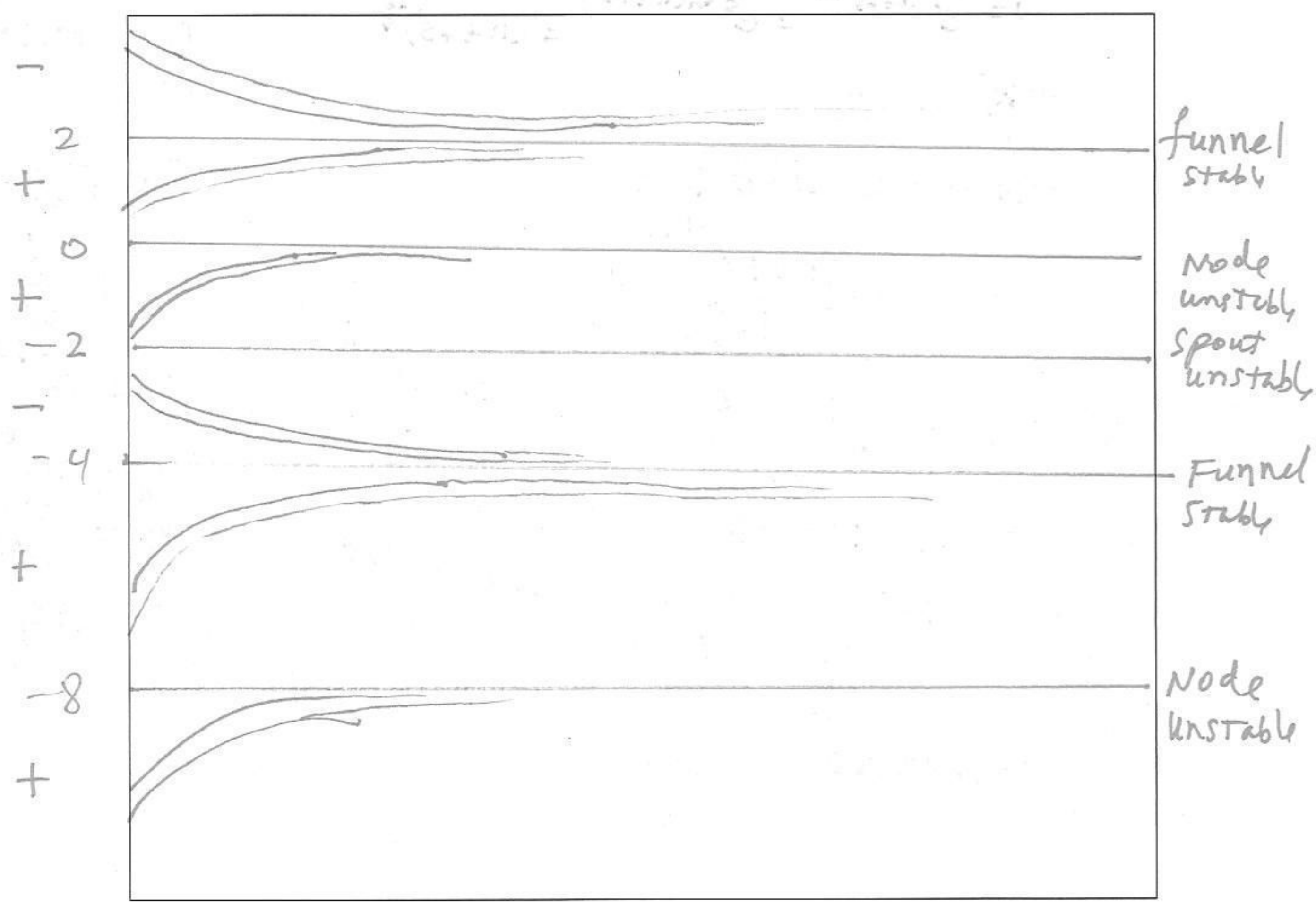
Expected in the phase line diagram are equilibrium points and signs of dx/dt. Definition: $\sinh(u) = \frac{1}{2}e^u - \frac{1}{2}e^{-u}$.

roots $x = -1, 1, 0, 3, -3, 2, -2$



(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.

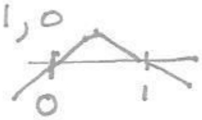




Calculations for problem 5 (a)

$\sinh(x+1) = 0 \Rightarrow x = -1$

$2 - |4x-2| = 0 \Rightarrow 4x - 2 = \pm 2 \Rightarrow x = \frac{1}{2} \pm \frac{1}{2} = 1, 0$



$3 - |x| = 0 \Rightarrow x = \pm 3$

$x^2 - 9 = 0 \Rightarrow x = \pm 3$

$4 - x^2 = 0 \Rightarrow x = \pm 2$

$x = 4:$ $+ \; - \; - \; + \; - \; = -$

$x = 2.5:$ $+ \; - \; + \; - \; - \; = -$
$= 5/2$

$x = 3/2:$ $+ \; - \; + \; - \; + \; = +$
$= 1.5$

$x = 1/2:$ $+ \; + \; + \; \ast \; + \; = -$

$x = -1/2:$ $+ \; - \; + \; - \; + \; = +$

$x = -3/2:$ $- \; - \; + \; - \; + \; = -$

$x = -5/2:$ $- \; - \; + \; - \; - \; = +$

$x = -4:$ $- \; - \; - \; + \; - \; = +$

$\sinh(-\frac{1}{2}+1) = \sinh(\frac{1}{2}) > 0$

$\sinh(-\frac{1}{2}) = \frac{e^{-1/2} - e^{1/2}}{2} < 0$