Electrical Circuits

- Voltage drop formulas of Faraday, Ohm, Coulomb.
- Kirchhoff’s laws.
- LRC Circuit equation.
- Electrical-Mechanical Analogy.
- Transient and Steady-state Currents.
- Reactance and Impedance.
- Time lag.
- Electrical Resonance.
Voltage Drop Formulas

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<th>Equation</th>
<th>Explanation</th>
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<td>Faraday’s Law</td>
<td>$V_L = L \frac{dI}{dt}$</td>
<td>$L = \text{inductance in henries,}$ $I = \text{current in amperes.}$</td>
</tr>
<tr>
<td>Ohm’s Law</td>
<td>$V_R = RI$</td>
<td>$R = \text{resistance in ohms.}$</td>
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<td>Coulomb’s Law</td>
<td>$V_C = \frac{Q}{C}$</td>
<td>$Q = \text{charge in coulombs,}$ $C = \text{capacitance in farads.}$</td>
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Kirchhoff’s Laws

The charge $Q$ and current $I$ are related by the equation

$$\frac{dQ}{dt} = I.$$

- **Loop Law**: The algebraic sum of the voltage drops around a closed loop is zero.
- **Junction Law**: The algebraic sum of the currents at a node is zero.
LRC Circuit Equation in Charge form

The first law of Kirchhoff implies the RLC circuit equation

\[ LQ'' + RQ' + \frac{1}{C}Q = E(t) \]

where inductor \( L \), resistor \( R \) and capacitor \( C \) are in a single loop having electromotive force \( E(t) \).

Figure 1. An LRC Circuit.
The components are a resistor \( R \), inductor \( L \), capacitor \( C \) and emf \( E(t) \). Current \( I(t) \) is assigned counterclockwise direction, from minus to plus on the emf terminals.
LRC Circuit Equation in Current Form

Differentiation of the charge form of the LRC circuit equation

\[ LQ'' + RQ' + \frac{1}{C}Q = E(t) \]

gives the current form of the LRC circuit equation

\[ LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}. \]
Electrical–Mechanical Analogy

\[ mx'' + cx' + kx = F(t), \]
\[ LQ'' + RQ + C^{-1}Q = E(t). \]

Table 1. Electrical–Mechanical Analogy

<table>
<thead>
<tr>
<th>Mechanical System</th>
<th>Electrical System</th>
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<tr>
<td>Mass ( m )</td>
<td>Inductance ( L )</td>
</tr>
<tr>
<td>Dampening constant ( c )</td>
<td>Resistance ( R )</td>
</tr>
<tr>
<td>Hooke’s constant ( k )</td>
<td>Reciprocal capacitance ( 1/C )</td>
</tr>
<tr>
<td>Position ( x )</td>
<td>Charge ( Q ) [or Current ( I )]</td>
</tr>
<tr>
<td>External force ( F )</td>
<td>Electromotive force ( E ) [or ( dE/dt )]</td>
</tr>
</tbody>
</table>
Transient and Steady-state Currents

The theory of mechanical systems leads to electrical results by applying the electrical-mechanical analogy to the LRC circuit equation in current form with $E(t) = E_0 \sin \omega t$. We assume $L$, $R$ and $C$ positive.

- The solution $I_h$ of the homogeneous equation $LI'' + RI' + \frac{1}{C}I = 0$ is a transient current, satisfying
  $$\lim_{t \to \infty} I_h(t) = 0.$$

- The non-homogeneous equation $LI'' + RI' + \frac{1}{C}I = E_0 \omega \cos \omega t$ has a unique periodic solution [steady-state current]
  $$I_{ss}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + S^2}}, \quad S \equiv \omega L - \frac{1}{\omega C}, \quad \tan \alpha = \frac{\omega RC}{1 - LC \omega^2}.$$

It is found by the method of undetermined coefficients.
Reactance and Impedance

Write

\[ I_{ss}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + S^2}} \]

as

\[ I_{ss}(t) = \frac{E_0}{Z} \cos(\omega t - \alpha) \]

where

\[ Z = \sqrt{R^2 + S^2} \] is called the **impedance**

\[ S = \omega L - \frac{1}{\omega C} \] is called the **reactance**.
**Time Lag**

The steady-state current $I_{ss}(t) \frac{E_0}{Z} \cos(\omega t - \alpha)$ can be written as a sine function using trigonometric identity $\cos(x - \pi/2) = \sin(x)$ with $\alpha = \delta + \pi/2$:

$$I_{ss}(t) = \frac{E_0}{Z} \sin(\omega t - \delta), \quad \tan \delta = \frac{LC\omega^2 - 1}{\omega RC} = \frac{S}{R}.$$  

Because the input is $E(t) = E_0\omega \sin(\omega t)$, then the **time lag** between the input voltage and the steady-state current is

$$\frac{\delta}{\omega} = \frac{\arctan(S/R)}{\omega} \text{ seconds.}$$
Resonance in an LRC circuit is defined only for sinusoidal inputs $E(t) = E_0 \sin(\omega t)$. Then the differential equation in current form is

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{E_0 \omega}{L} \cos(\omega t).$$

Resonance happens if there is a frequency $\omega$ which maximizes the steady-state solution amplitude $I_0 = E_0/Z$, $Z = \sqrt{R^2 + S^2}$, $S = \omega L - \frac{1}{C \omega}$. By calculus, this happens exactly when $dZ/d\omega = 0$, which gives the resonant frequency

$$\omega = \frac{1}{\sqrt{LC}}.$$

Details: $dI_0/d\omega = 0$ if and only if $-E_0 Z^{-2} dZ/d\omega = 0$, which is equivalent to $dZ/d\omega = 0$. Then $2S dS/d\omega = 0$ and finally $S = 0$, because $dS/d\omega > 0$. The equation $S = 0$ is equivalent to $\omega = 1/\sqrt{LC}$. 

Electrical Resonance