How to Solve Linear Differential Equations

- Definition: Euler Base Atom, Euler Solution Atom
- Independence of Atoms
- Construction of the General Solution from a List of Distinct Atoms
- Euler’s Theorems
  - Euler’s Basic Theorem
  - Euler’s Multiplicity Theorem
  - A Shortcut Method
- Examples
- Main Theorems on Atoms and Linear Differential Equations
Euler Solution Atoms of Homogeneous Linear Differential Equations

Definition

• An Euler base atom is one of $1$, $\cos bx$, $\sin bx$ with $b > 0$, or one of $e^{ax}$, $e^{ax} \cos bx$, $e^{ax} \sin bx$, with $a \neq 0$ (multiply the first three by $e^{ax}$).

• An Euler solution atom equals a base atom, or a base atom multiplied by one of the integer powers $x, x^2, \ldots$ (positive integer powers only).

Details and Remarks

• Euler’s formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies that an atom is constructed from the complex expression $x^n e^{ax+ibx}$ by taking real and imaginary parts.

• The powers $1, x, x^2, \ldots, x^k$ are Euler solution atoms.

• The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom $e^x$. Zero is not an atom. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms $x$ and $x^2$ are indeed atoms. Terms like $-e^x$, $e^{-x^2}$, $x^{5/2} \cos x$, $\ln |x|$ and $x/(1 + x^2)$ are not atoms.
Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$$

for all $x$ implies $c_1 = c_2 = \cdots = c_k = 0$.

A **function** is a data package consisting of an equation $y = f(x)$ and an $x$-domain $D$. The vector notation is $\vec{f}$ (no visible equation or domain). The scalar notation is $f$ or $f(x)$.

Independence and Atoms

**Theorem 1 (Atoms are Independent)**

A list of finitely many distinct Euler solution atoms is linearly independent.

**Theorem 2 (Powers are Independent)**

The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.
Construction of the General Solution from a List of Distinct Atoms 

- **Picard’s theorem** says that the homogeneous constant-coefficient linear differential equation 

  \[ y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y = 0 \]

  has solution space \( S \) of dimension \( n \). Picard’s theorem reduces the general solution problem to finding \( n \) linearly independent solutions.

- **Euler’s theorem** *infra* says that the required \( n \) independent solutions can be selected as atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [Definition: The left side is the characteristic polynomial.]

  \[ r^n + p_{n-1}r^{n-1} + \cdots + p_1r + p_0 = 0. \]

- The **Fundamental Theorem of Algebra** is part of the Doctoral thesis of Carl Friedrich Gauss (1777–1855): An \( n \)th order polynomial equation has exactly \( n \) roots, real or complex, counted according to multiplicities. Therefore, the characteristic equation has exactly \( n \) roots, counting multiplicities.

- **General Solution.** Because the list of atoms constructed by Euler’s theorem has \( n \) distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

  \[ y = c_1(\text{atom 1}) + \cdots + c_n(\text{atom } n). \]

  Symbols \( c_1, \ldots, c_n \) are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation. The **solution space** of the differential equation is \( S = \text{span}(\text{the } n \text{ atoms}) \).
Euler’s Basic Theorem

Theorem 3 (L. Euler)
The exponential $y = e^{r_1x}$ is a solution of a constant-coefficient linear homogeneous differential of the $n$th order if and only if $r = r_1$ is a root of the characteristic equation.

- If $r_1 = a$ is a real root, then Euler’s Theorem constructs one real solution atom $e^{ax}$.
- If $r_1 = a + ib$ is a complex root ($b > 0$), then Euler’s Theorem constructs two real solution atoms
  \[ e^{ax} \cos bx, \quad e^{ax} \sin bx. \]

Derivation is from Euler’s complex solution
\[ e^{r_1x} = e^{ax} \cos bx + ie^{ax} \sin bx. \]

The real and imaginary parts of this complex solution are real solutions of the differential equation. The conjugate pair of roots $a + ib$ and $a - ib$ produce the same two atoms, which explains the simplification above.
Euler’s Multiplicity Theorem

**Definition.** A root \( r = r_1 \) of a polynomial equation \( p(r) = 0 \) has **multiplicity** \( k \) provided \((r - r_1)^k \) divides \( p(r) \) but \((r - r_1)^{k+1} \) does not divide \( p(r) \). The calculus equivalent is \( \frac{d^j p}{dr^j}(r_1) = 0 \) for \( j = 0, \ldots, k - 1 \) and \( \frac{d^k p}{dr^k}(r_1) \neq 0 \).

**Theorem 4 (L. Euler)**

The expression \( y = x^k e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \((r - r_1)^{k+1} \) divides the characteristic polynomial.

**A Shortcut for using Euler’s Theorems**

Given a real root \( r_1 \) or complex root \( r_1 = a + ib \), apply Euler’s first theorem to obtain the base atom \( e^{r_1 x} \), or the pair of base atoms \( e^{ax} \cos bx, e^{ax} \sin bx \). Multiply each base item by powers 1, \( x \), \( x^2 \), \ldots, until the number of atoms obtained equals the multiplicity of root \( r_1 \).
Atom List Examples

1. If root $r = -3$ has multiplicity 4, then the atom list is

$$e^{-3x}, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}.$$  

The list is constructed by multiplying the base atom $e^{-3x}$ by powers 1, $x$, $x^2$, $x^3$. The multiplicity 4 of the root equals the number of constructed atoms.

2. If $r = -3 + 2i$ is a root of the characteristic equation, then the base atoms for this root (both $-3 + 2i$ and $-3 - 2i$ counted) are

$$e^{-3x} \cos 2x, e^{-3x} \sin 2x.$$  

If root $r = -3 + 2i$ has multiplicity 3, then the two real atoms are multiplied by 1, $x$, $x^2$ to obtain a total of 6 atoms

$$e^{-3x} \cos 2x, xe^{-3x} \cos 2x, x^2e^{-3x} \cos 2x, e^{-3x} \sin 2x, xe^{-3x} \sin 2x, x^2e^{-3x} \sin 2x.$$  

The number of atoms generated for each base atom is 3, which equals the multiplicity of the root $-3 + 2i$.  

Theorem 5 (Homogeneous Solution $y_h$ and Euler Solution Atoms)
Linear homogeneous $n$th order differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of $n$ distinct atoms.

Theorem 6 (Particular Solution $y_p$ and Euler Solution Atoms)
A linear non-homogeneous differential equation with constant coefficients having a forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 7 (General Solution $y$ and Euler Solution Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{a linear combination of atoms}.$$ 

Proofs
The first theorem follows from Picard’s theorem, Euler’s theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.