Special Topics from Asmar's Textbook, Chapter 1

- The Wronskian Determinant
- Quadrature, Arbitrary Constants and Arbitrary Functions
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The Wronskian Determinant

Definition. The Wronskian Matrix of two functions $f_1(x)$, $f_2(x)$ is

$$W(x) = \begin{pmatrix} f_1(x) & f_2(x) \\ \frac{d}{dx} f_1(x) & \frac{d}{dx} f_2(x) \end{pmatrix}.$$

The Wronskian Determinant of two functions $f_1(x)$, $f_2(x)$ is $\det(W(x))$. The determinant of a 2×2 matrix is defined by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

1 **Example (Compute a Wronskian Determinant)** Find the Wronskian determinant of the two function x^2, x^5. Answer:

$$W(x) = \begin{vmatrix} x^2 & x^5 \\ 2x & 5x^4 \end{vmatrix} = 3x^6.$$

The Pattern: For the Wronskian matrix of n functions $f_1, \ldots, f_n$, construct the first row of $W(x)$ as the n values $f_1(x)$ to $f_n(x)$. Then differentiate row 1 successively to obtain the other rows of $W(x)$. The last row is $\frac{d^{n-1}}{dx^{n-1}}$ applied to row 1.

Quadrature, Arbitrary Constants and Arbitrary Functions

The **linear ordinary differential equation** $\boldsymbol{y'' = -32}$ has general solution $\boldsymbol{y(x) = -16x^2 + c_1 x + c_2}$, where $\boldsymbol{c_1, c_2}$ are arbitrary constants. This is typical:

> *The order of a linear ordinary differential equation determines the number of arbitrary* **constants** *in the general solution.*

The analog for **partial differential equations** is this:

> *The order of the partial differential equation determines the number of arbitrary* **functions** *appearing in the general solution.*

Theorem 1 (Quadrature for Partial Differential Equations)
Let $u(x, y)$ satisfy the partial differential equation

$$\frac{\partial u}{\partial x} = 0.$$

Then $u(x, y) = f(y)$ where f is an arbitrary function of one variable.

Proof: Apply the method of quadrature to the equation $\dfrac{\partial u}{\partial x} = 0$, as follows:

$$\begin{array}{ll}
\displaystyle\int_0^x \frac{\partial u(x,y)}{\partial x} dx = \int_0^x 0 dx & \text{Multiply by } dx \text{ and integrate} \\
u(x,y) - u(0,y) = 0 & \text{Fundamental Theorem of Calculus} \\
u(x,y) = u(0,y) & \text{Function } u(0,y) \text{ depends only on } y \\
u(x,y) = f(y) & \text{Where } f \text{ is an arbitrary function.}
\end{array}$$

Remark. In general, u is an arbitrary function of all variables other than x.

Application: Change of variables

We'll solve the advection equation $u_t + 15u_x = 0$ by an invertible change of variables $r = at + bx$, $s = ct + dx$. The answer is $u = f(x - 15t)$ where $f(w)$ is an arbitrary differentiable real-valued function of scalar variable w.

The Plan. The change of variables transforms (t, x) into (r, s), to obtain the new differential equation $\partial u/\partial r = 0$. Then u is a constant for each fixed s, hence $u = f(s)$ for some arbitrary function f.

Details. Compute u_t by the chain rule, then $u_t = u_r r_t + u_s s_t = a u_r + c u_s$. Similarly, $u_x = b u_r + d u_s$. Then $u_t + 15u_x = 0$ becomes upon substitution the new equation $(a + 15b)u_r + (c + 15d)u_s = 0$. The choices $a + 15b = 1$ and $c + 15d = 0$ will make the new equation into $u_r = 0$, as required. The constants a, b, c, d are selected as $a = -14, b = 1, c = -15, d = 1$ in order to make the change of variables invertible (nonzero determinant). Then $s = -15t + x$ and $u = f(s) = f(x - 15t)$.

Making a Filmstrip with Maple: The Advection Equation

Consider $\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$, $u(0,t) = e^{-2t^2}$. The solution is easily checked to be $u(t,x) = e^{-2(x-2t)^2}$. We will make a filmstrip of 5 graphics at $x = 0, 1, 2, 3, 4$. Each graphic is a plot of t against u on interval $-1 < t < 5$.

```
u:=(x,t)->exp(-2*(x-2*t)^2);
mycolor:=[black,red,yellow,orange,green]:
xval:=[0,1,2,3,4]:
myplots:=[seq(plot(u(xval[i],t),t=-1..2,color=mycolor[i]),i = 1..5)]:
plots[display](myplots,insequence=true); # Animation
for i from 1 to 5 do myplots[i]; end do; # Make 5 individual plots
```

Method of Characteristics

Definition. A first order partial differential equation

$$v_1(x,y)\frac{\partial u(x,y)}{\partial x} + v_2(x,y)\frac{\partial u(x,y)}{\partial y} = 0 \tag{1}$$

has **characteristic curves** defined by the implicit solution

$$w(x,y) = c$$

of the associated **characteristic differential equation**

$$-v_2(x,y)dx + v_1(x,y)dy = 0.$$

Theorem 2 (General Solution)
Let $f(w)$ denote an arbitrary function. Then the general solution of (1) is given by

$$u(x,y) = f(w(x,y)).$$

Application: the Method of Characteristics

We solve the equation $-xu_x + yu_y = 0$ by the method of characteristics. The answer is $u = f(xy)$ where $f(w)$ is a real-valued arbitrary differentiable function of scalar variable w.

Solution: First we construct the characteristic equation, by the formal replacement process $u_x \to -dy$ and $u_y \to dx$. The ODE is $-x(-dy) + ydx = 0$ or equivalently $y' = -y/x$. This is a first order linear homogeneous ODE with solution $y =$ constant/integrating factor $= c/x$. We solve $y = c/x$ for c to get the implicit equation $xy = c$. Then $w(x, y) = xy$ in the Theorem (see the previous slide) and we have general solution $u = f(w(x, y))$, reported as $u = f(xy)$.

Answer check: Compute LHS $= -xu_x + yu_y = -x\partial_x(f(xy)) + y\partial_y(f(xy)) = -xf'(xy)y + yf'(xy)x = 0$, and RHS $= 0$, therefore LHS $=$ RHS for all symbols.

General Solution by the Method of Characteristics: The Proof

Proof: Let $f(w)$ denote an arbitrary function. We prove that the general solution of (1) is given by $u(x,y) = f(w(x,y))$. First, suppose that (x_0, y_0) is a point of the characteristic curve $w(x,y) = c$ and y is locally determined as a function of x, e.g., $v_1(x,y) \neq 0$ and $y = y(x)$. Then $y(x)$ is differentiable and $y' = v_2/v_1$. Assume $u(x,y)$ is a solution of (1), then we compute

$$\begin{aligned}
\frac{d}{dx}u(x,y(x)) &= \frac{\partial u}{\partial x} + y'(x)\frac{\partial u}{\partial y} \\
&= \frac{1}{v_1(x,y)}\left(v_1(x,y)\frac{\partial u}{\partial x} + v_2(x,y)\frac{\partial u}{\partial y}\right) \\
&= 0.
\end{aligned}$$

If the derivative is zero, then $u(x,y(x))$ must be a constant which depends only on (x_0, y_0), or ultimately on the constant c in the equation $w(x_0,y_0) = c$. Therefore, $u(x,y(x)) = f(c)$ for some function $f(w)$. Using the implicit solution, then $u(x,y(x)) = f(c) = f(w(x,y(x)))$ or simply $u(x,y) = f(w(x,y))$. The proof is completed by showing directly that this solution satisfies the partial differential equation.

d'Alembert's Solution to the Wave Equation

The wave equation for an infinite string is $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, where $-\infty < x < \infty$ and $t \geq 0$ is time.

Theorem 3 (d'Alembert's Solution)
The infinite string equation has general solution

$$u(x,t) = F(x+ct) + G(x-ct)$$

where F and G are twice continuously differentiable functions of one variable.

Proof: The change of variables $r = x + ct$, $s = x - ct$ from (x,t) into (r,s) implies the partial differential equation $\frac{\partial}{\partial s}\frac{\partial}{\partial r} u((r+s)/2, (r-s)/(2c)) = 0$. This equation is solved by quadrature to obtain the result.

Application: d'Alembert's Solution

We solve the wave equation for an infinite string, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, where $-\infty < x < \infty$ and $t \geq 0$ is time. The initial conditions are $u(x,0) = \frac{1}{4+x^2}$, $u_t(x,0) = 0$.

Solution. The method is d'Alembert's solution $u(x,t) = F(x+ct) + G(x-ct)$ where F and G are twice continuously differentiable functions of one variable. Let $h(x) = u(x,0) = \frac{1}{4+x^2}$. We get from setting $t=0$ in the conditions the two equations $F(x)+G(x) = h(x)$, $cF'(x) - cG'(x) = 0$. The second equation implies $G(x) = F(x) + d$ for some constant d. Then $F(x)+F(x)+d = u(x,0)$ determines F. Re-label $f(x) = F(x) + d/2$. Then $F(x)+G(x) = f(x) - d/2 + f(x) + d/2 = 2f(x)$, or $f(x) = (1/2)h(x)$. Finally, $u(x,t) = f(x+ct) - d/2 + f(x-ct) + d/2 = f(x+ct) + f(x-ct)$. Then

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(h(x+ct) + h(x-ct)) \\
&= \frac{1/2}{4+(x+ct)^2} + \frac{1/2}{4+(x-ct)^2}.
\end{aligned}
$$