Chapter 12

Series Methods and Approximations

Contents

12.1 Review of Calculus Topics **699**
12.2 Algebraic Techniques **701**
12.3 Power Series Methods **707**
12.4 Ordinary Points **712**
12.5 Regular Singular Points **715**
12.6 Bessel Functions **727**
12.7 Legendre Polynomials **731**
12.8 Orthogonality **738**

The differential equation

$$(1+x^2)y'' + (1+x+x^2+x^3)y' + (x^3-1)y = 0 \tag{1}$$

has polynomial coefficients. While it is not true that such differential equations have polynomial solutions, it will be shown in this chapter that for graphical purposes it is **almost true**: the general solution y can be written as

$$y(x) \approx c_1 p_1(x) + c_2 p_2(x),$$

where p_1 and p_2 are polynomials, which depend on the graph window, pixel resolution and a maximum value for $|c_1| + |c_2|$.

In particular, graphs of solutions can be made with a graphing hand calculator, a computer algebra system or a numerical laboratory by entering two polynomials p_1, p_2. For (1), the polynomials

$$p_1(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5,$$

$$p_2(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{15}x^5$$

can be used to plot solutions within a reasonable range of initial conditions.

The theory will show that (1) has a basis of solutions $y_1(x)$, $y_2(x)$, each represented as a convergent power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Truncation of the two power series gives two polynomials p_1, p_2 (approximate solutions) suitable for graphing solutions of the differential equation by the approximation formula $y(x) \approx c_1 p_1(x) + c_2 p_2(x)$.

12.1 Review of Calculus Topics

A **power series** in the variable x is a formal sum

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots. \tag{2}$$

It is called **convergent** at x provided the limit below exists:

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n x^n = L.$$

The value L is a finite number called the **sum** of the series, written usually as $L = \sum_{n=0}^{\infty} c_n x^n$. Otherwise, the power series is called **divergent**. Convergence of the power series for every x in some interval J is called **convergence on** J. Similarly, **divergence on** J means the power series fails to have a limit at each point x of J. The series is said to **converge absolutely** if the series of absolute values $\sum_{n=0}^{\infty} |c_n||x|^n$ converges.

Given a power series $\sum_{n=0}^{\infty} c_n x^n$, define the **radius of convergence** R by the equation

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|. \tag{3}$$

The radius of convergence R is undefined if the limit does not exist.

Theorem 1 (Maclaurin Expansion)
If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and $R > 0$, then f has infinitely many derivatives on $|x| < R$ and its coefficients $\{c_n\}$ are given by the **Maclaurin formula**

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{4}$$

The example $f(x) = e^{-1/x^2}$ shows the theorem has no converse. The following basic result summarizes what appears in typical calculus texts.

Theorem 2 (Convergence of power series)
Let the power series $\sum_{n=0}^{\infty} c_n x^n$ have radius of convergence R. If $R = 0$, then the series converges for $x = 0$ only. If $R = \infty$, then the series converges for all x. If $0 < R < \infty$, then

1. The series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely if $|x| < R$.
2. The series $\sum_{n=0}^{\infty} c_n x^n$ diverges if $|x| > R$.
3. The series $\sum_{n=0}^{\infty} c_n x^n$ may converge or diverge if $|x| = R$. The **interval of convergence** may be of the form $-R < x < R$, $-R \le x < R$, $-R < x \le R$ or $-R \le x \le R$.

Library of Maclaurin Series. Below we record the key Maclaurin series formulas used in applications.

Geometric Series: $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$	Converges for $-1 < x < 1$.
Log Series: $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	Converges for $-1 < x \le 1$.
Exponential Series: $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	Converges for all x.
Cosine Series: $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	Converges for all x.
Sine Series: $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	Converges for all x.

Theorem 3 (Properties of power series)
Given two power series $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$ with radii of convergence R_1, R_2, respectively, define $R = \min(R_1, R_2)$, so that both series converge for $|x| < R$. The power series have these properties:

1. $\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$ implies $b_n = c_n$ for all n.
3. $\sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (b_n + c_n) x^n$ for $|x| < R$.
4. $k \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} k b_n x^n$ for all constants k, $|x| < R_1$.
5. $\frac{d}{dx} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} n b_n x^{n-1}$ for $|x| < R_1$.
6. $\int_a^b \left(\sum_{n=0}^{\infty} b_n x^n\right) dx = \sum_{n=0}^{\infty} b_n \int_a^b x^n dx$ for $-R_1 < a < b < R_1$.

Taylor Series. A series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called a **Taylor series** expansion of $f(x)$ about $x = x_0$. If valid, then the series converges and represents $f(x)$ for an interval of convergence $|x - x_0| < R$. Taylor expansions are general-use extensions of Maclaurin expansions, obtained by translation $x \to x - x_0$. If a Taylor series exists, then $f(x)$ has infinitely many derivatives. Therefore, $|x|$ and x^α $(0 < \alpha < 1)$ fail to have Taylor expansions about $x = 0$. On the other hand, e^{-1/x^2} has infinitely many derivatives, but no Taylor expansion at $x = 0$.

12.2 Algebraic Techniques

Derivative Formulas. Differential equations are solved with series techniques by assuming a **trial solution** of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

The trial solution is thought to have **undetermined coefficients** $\{c_n\}$, to be found explicitly by the method of undetermined coefficients, i.e., substitute the trial solution and its derivatives into the differential equation and resolve the constants. The various derivatives of $y(x)$ can be written as power series. Recorded here are the mostly commonly used derivative formulas.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n (x - x_0)^n, \\
y'(x) &= \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}, \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x - x_0)^{n-2}, \\
y'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) c_n (x - x_0)^{n-3}.
\end{aligned}$$

The summations are over a different subscript range in each case, because differentiation eliminates the constant term each time it is applied.

Changing Subscripts. A change of variable $t = x - a$ changes an integral $\int_a^\infty f(x)dx$ into $\int_0^\infty f(t + a)dt$. This change of variable is indicated when several integrals are added, because then the interval of integration is $[0, \infty)$, allowing the various integrals to be collected on one integral sign. For instance,

$$\int_2^\infty f(x)dx + \int_\pi^\infty g(x)dx = \int_0^\infty (f(t+2) + g(t+\pi))dt.$$

A similar change of variable technique is possible for summations, allowing several summation signs with different limits of summation to be collected under one summation sign. The rule:

$$\sum_{n=a}^{n=a+h} x_n = \sum_{k=0}^{h} x_{k+a}.$$

It is remembered via the change of variable $k = n - a$, which is formally applied to the summation just as it is applied in integration theory. If $h = \infty$, then the rule reads as follows:

$$\sum_{n=a}^{\infty} x_n = \sum_{k=0}^{\infty} x_{k+a}.$$

An illustration, in which LHS refers to the substitution of a trial solution into the left hand side of some differential equation,

$$\begin{aligned}
\text{LHS} &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2x\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{n=0}^{\infty} 2c_n x^{n+1} \\
&= 2c_0 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} 2c_{k-1}x^k \\
&= 2c_0 + \sum_{k=1}^{\infty}((k+2)(k+1)c_{k+2} + 2c_{k-1})x^k.
\end{aligned}$$

To document the steps: **Step 1** is the result of substitution of the trial solution into the differential equation $y'' + 2xy$; **Step 2** makes a change of index variable $k = n - 2$; **Step 3** makes a change of index variable $k = n + 1$; **Step 4** adds the two series, which now have the same range of summation and equal powers of x. The change of index variable in each case was dictated by attempting to match the powers of x, e.g., $x^{n-2} = x^k$ in **Step 2** and $x^{n+1} = x^k$ in **Step 3**.

The formulas for derivatives a trial solution $y(x)$ can all be written with the same index of summation, if desired:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n (x-x_0)^n, \\
y'(x) &= \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-x_0)^n, \\
y''(x) &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} (x-x_0)^n, \\
y'''(x) &= \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) c_{n+3} (x-x_0)^n.
\end{aligned}$$

Linearity and Power Series. The set of all power series convergent for $|x| < R$ form a vector space under function addition and scalar multiplication. This means:

1. The sum of two power series is a power series.

2. A scalar multiple of a power series is a power series.

3. The zero power series is the zero function: all coefficients are zero.

4. The negative of a power series is (-1) times the power series.

Cauchy Product. Multiplication and division of power series is possible and the result is again a power series convergent on some interval $|x| < R$. The **Cauchy product** of two series is defined by the relations

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{k=0}^{\infty} c_k x^k, \quad c_k = \sum_{n=0}^{k} a_n b_{k-n}.$$

Division of two series can be defined by its equivalent Cauchy product formula, which determines the coefficients of the quotient series.

To illustrate, we compute the coefficients $\{c_n\}$ in the formula

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{k=0}^{\infty} \frac{x^k}{k+1}\right) \Big/ \left(\sum_{m=0}^{\infty} x^m\right).$$

Limitations exist: the division is allowed only when the denominator is nonzero. In the present example, the denominator sums to $1/(1-x)$, which is never zero. The equivalent Cauchy product relation is

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{m=0}^{\infty} x^m\right) = \sum_{k=0}^{\infty} \frac{x^k}{k+1}.$$

This relation implies the formula

$$\sum_{n=0}^{k}(c_n)(1) = \frac{1}{k+1}.$$

Therefore, back-substitution implies $c_0 = 1$, $c_1 = -1/2$, $c_2 = -1/6$. More coefficients can be found and perhaps also a general formula can be written for c_n. Such a formula is needed infrequently, so we spend no time discussing how to find it.

Power Series Expansions of Rational Functions. A rational function $f(x)$ is a quotient of two polynomials, therefore it is a quotient of two power series, hence also a power series. Sometimes the easiest method known to find the coefficients c_n of the power series of f is to apply Maclaurin's formula

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

In a number of limited cases, in which the polynomials have low degree, it is possible to use Cauchy's product formula to find $\{c_n\}$. An illustration:

$$\frac{x+1}{x^2+1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{2k+1} = c_{2k} = (-1)^k.$$

To derive this formula, write the quotient as a Cauchy product:

$$\begin{aligned}
x+1 &= (1+x^2)\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} c_n x^n + \sum_{m=0}^{\infty} c_m x^{m+2} \\
&= c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n + \sum_{k=2}^{\infty} c_{k-2} x^k \\
&= c_0 + c_1 x + \sum_{k=2}^{\infty} (c_k + c_{k-2}) x^k
\end{aligned}$$

The third step uses variable change $k = m + 2$. The series then have the same index range, allowing the addition of the final step. To match coefficients on each side of the equation, we require $c_0 = 1$, $c_1 = 1$, $c_k + c_{k-2} = 0$. Solving, $c_2 = -c_0$, $c_3 = -c_1$, $c_4 = -c_2 = (-1)^2 c_0$, $c_5 = -c_3 = (-1)^2 c_1$. By induction, $c_{2k} = (-1)^k$ and $c_{2k+1} = (-1)^k$. This gives the series reported earlier.

The same series expansion can be obtained in a more intuitive manner, as follows. The idea depends upon substitution of $r = -x^2$ into the geometric series expansion $(1-r)^{-1} = 1 + r + r^2 + \cdots$, which is valid for $|r| < 1$.

$$\begin{aligned}
\frac{x+1}{x^2+1} &= (1+x)\sum_{n=0}^{\infty} r^n \quad \text{where } r = -x^2 \\
&= \sum_{n=0}^{\infty}(-x^2)^n + x\sum_{n=0}^{\infty}(-x^2)^n \\
&= \sum_{n=0}^{\infty}(-1)^n x^{2n} + \sum_{n=0}^{\infty}(-1)^n x^{2n+1} \\
&= \sum_{k=0}^{\infty} c_k x^k,
\end{aligned}$$

where $c_{2k} = (-1)^k$ and $c_{2k+1} = (-1)^k$. The latter method is preferred to discover a useful formula. The method is a shortcut to the expansion of $1/(x^2+1)$ as a Maclaurin series, followed by series properties to write the indicated Cauchy product as a single power series.

Instances exist where neither the Cauchy product method nor other methods are easy, for instance, the expansion of $f(x) = 1/(x^2+x+1)$. Here, we might find a formula from $c_n = f^{(n)}(0)/n!$, or equally unpleasant, find $\{c_n\}$ from the formula $1 = (x^2+x+1)\sum_{n=0}^{\infty} c_n x^n$.

Recursion Relations. The relations

$$c_0 = 1, \quad c_1 = 1, \quad c_k + c_{k-2} = 0 \text{ for } k \geq 2$$

are called **recursion relations**. They are often solved by ad hoc algebraic methods. Developed here is a systematic method for solving such recursions.

First order recursions. Given x_0 and sequences of constants $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, consider the abstract problem of finding a formula for x_k in the recursion relation

$$x_{k+1} = a_k x_k + b_k, \quad k \geq 0.$$

For $k = 0$ the formula gives $x_1 = a_0 x_0 + b_0$. Similarly, $x_2 = a_1 x_1 + b_1 = a_1 a_0 x_0 + a_1 b_0 + b_1$, $x_3 = a_2 x_2 + b_2 = a_2 a_1 a_0 x_0 + a_2 a_1 b_0 + a_2 b_1 + b_2$. By induction, the unique solution is

$$x_{k+1} = \left(\Pi_{r=0}^{k} a_r\right) x_0 + \sum_{n=0}^{k}\left(\Pi_{r=n+1}^{k} a_r\right) b_n.$$

Two-termed second order recursions. Given c_0, c_1 and sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$, consider the problem of solving for c_{k+2} in the two-termed second order recursion

$$c_{k+2} = a_k c_k + b_k, \quad k \geq 0.$$

The idea to solve it comes from splitting the problem into even and odd subscripts. For even subscripts, let $k = 2n$. For odd subscripts, let $k = 2n + 1$. Then the two-termed second order recursion splits into two first order recursions

$$\begin{aligned} c_{2n+2} &= a_{2n}c_{2n} + b_{2n}, \quad n \geq 0, \\ c_{2n+3} &= a_{2n+1}c_{2n+1} + b_{2n+1}, \quad n \geq 0. \end{aligned}$$

Define $x_n = c_{2n}$ or $x_n = c_{2n+1}$ and apply the general theory for first order recursions to solve the above recursions:

$$\begin{aligned} c_{2n+2} &= \left(\Pi_{r=0}^{n} a_{2r}\right) c_0 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{2r}\right) b_{2r}, \quad n \geq 0, \\ c_{2n+3} &= \left(\Pi_{r=0}^{n} a_{2r+1}\right) c_1 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{2r+1}\right) b_{2r+1}, \quad n \geq 0. \end{aligned}$$

Two-termed third order recursions. Given c_0, c_1, c_2, $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$, consider the problem of solving for c_{k+3} in the two-termed third order recursion

$$c_{k+3} = a_k c_k + b_k, \quad k \geq 0.$$

The subscripts are split into three groups by the equations $k = 3n$, $k = 3n + 1$, $k = 3n + 2$. Then the third order recursion splits into three first order recursions, each of which is solved by the theory of first order recursions. The solution for $n \geq 0$:

$$\begin{aligned} c_{3n+3} &= \left(\Pi_{r=0}^{n} a_{3r}\right) c_0 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{3r}\right) b_{3r}, \\ c_{3n+4} &= \left(\Pi_{r=0}^{n} a_{3r+1}\right) c_1 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{3r+1}\right) b_{3r+1}, \\ c_{3n+5} &= \left(\Pi_{r=0}^{n} a_{3r+2}\right) c_2 + \sum_{k=0}^{n} \left(\Pi_{r=k+1}^{n} a_{3r+2}\right) b_{3r+2}. \end{aligned}$$

12.3 Power Series Methods

A Series Method for First Order. Illustrated here is a method to solve the differential equation $y' - 2y = 0$ for a power series solution. Assume a power series **trial solution**

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Let LHS stand for the left hand side of $y' - 2y = 0$. Substitute the trial series solution into the left side to obtain:

$$
\begin{aligned}
\text{(1)} \qquad \text{LHS} &= y' - 2y \\
&= \sum_{n=1}^{\infty} n c_n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k + \sum_{n=0}^{\infty} (-2) c_n x^n \\
&= \sum_{k=0}^{\infty} ((k+1) c_{k+1} - 2c_k) x^k
\end{aligned}
$$

(2)

The change of variable $k = n-1$ was used in the third step, the objective being to add on like powers of x. Because LHS $= 0$, and the zero function is represented by the series of all zero coefficients, then all coefficients in the series for LHS must be zero, which gives the recursion relation

$$(k+1)c_{k+1} - 2c_k = 0, \quad k \geq 0.$$

This first order two-termed recursion is solved by back-substitution or by using the general theory for first order recursions which appears above. Then

$$
\begin{aligned}
c_{k+1} &= \left(\Pi_{r=0}^{k} \frac{2}{r+1} \right) c_0 \\
&= \frac{2^{k+1}}{(k+1)!} c_0.
\end{aligned}
$$

The **trial solution** becomes a **power series solution**:

$$
\begin{aligned}
y(x) &= c_0 + \sum_{k=0}^{\infty} c_{k+1} x^{k+1} && \text{Re-index the trial solution.} \\
&= c_0 + \sum_{k=0}^{\infty} \frac{2^{k+1}}{(k+1)!} c_0 \, x^{k+1} && \text{Substitute the recursion answer.} \\
&= c_0 + \left(\sum_{n=1}^{\infty} \frac{2^n}{(n)!} x^n \right) c_0 && \text{Change index } n = k+1.
\end{aligned}
$$

$$= \left(\sum_{n=0}^{\infty} \frac{(2x)^n}{(n)!}\right) c_0 \qquad \text{Compress } c_0 \text{ into sum.}$$
$$= e^{2x} c_0. \qquad \text{Maclaurin expansion library.}$$

The **solution** $y(x) = c_0 e^{2x}$ agrees with the growth-decay theory formula for the first order differential equation $y' = ky$ ($k = 2$ in this case).

A Series Method for Second Order. Shown here are the details for finding two independent power series solutions

$$y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \frac{1}{1710720}x^{12} + \cdots$$
$$y_2(x) = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45360}x^{10} + \frac{1}{7076160}x^{13} + \cdots$$

for Airy's airfoil differential equation

$$y'' = xy.$$

The two independent solutions give the general solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

The solutions are related to the classical **Airy wave functions**, denoted `AiryAi` and `AiryBi` in the literature, and documented for example in the computer algebra system `maple`. The wave functions `AiryAi`, `AiryBi` are special linear combinations of y_1, y_2.

The **trial solution** in the second order power series method is generally a Taylor series. In this case, it is a Maclaurin series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Write Airy's differential equation in standard form $y'' - xy = 0$ and let LHS stand for the left hand side of this equation. Then substitution of the trial solution into LHS gives:

$$\begin{aligned}
\text{LHS} &= y'' - xy \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - x\sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{k=0}^{\infty} c_k x^{k+1} \\
&= 2c_2 + \sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n \\
&= 2c_2 + \sum_{n=1}^{\infty}\left((n+2)(n+1)c_{n+2} - c_{n-1}\right)x^n
\end{aligned}$$

The **steps**: (1) Substitute the trial solution into LHS using derivative formulas; (2) Move x inside the summation by linearity; (3) Index change $n = k+1$ to match powers of x; (4) Match summation index ranges and collect on powers of x.

Because LHS $= 0 =$ RHS and the power series for the zero function has zero coefficients, all coefficients in the series LHS must be zero. This implies the relations

$$c_2 = 0, \quad (n+2)(n+1)c_{n+2} - c_{n-1} = 0, \quad n \geq 1.$$

Replace n by $k+1$. Then the relations above become the two-termed third order recursion

$$c_{k+3} = \frac{1}{(k+2)(k+3)}\, c_k, \quad k \geq 0.$$

The answers from page 706, taking $b_k = 0$:

$$\begin{aligned}
c_{3n+3} &= \left(\Pi_{r=0}^{n} \frac{1}{(3r+2)(3r+3)}\right) c_0, \\
c_{3n+4} &= \left(\Pi_{r=0}^{n} \frac{1}{(3r+3)(3r+4)}\right) c_1, \\
c_{3n+5} &= \left(\Pi_{r=0}^{n} \frac{1}{(3r+4)(3r+5)}\right) c_2 \\
&= 0 \quad \text{(because } c_2 = 0\text{)}.
\end{aligned}$$

Taking $c_0 = 1$, $c_1 = 0$ gives one solution

$$y_1(x) = 1 + \sum_{n=0}^{\infty} \left(\Pi_{r=0}^{n} \frac{1}{(3r+2)(3r+3)}\right) x^{3n+3}.$$

Taking $c_0 = 0$, $c_1 = 1$ gives a second independent solution

$$\begin{aligned}
y_2(x) &= x + \sum_{n=0}^{\infty} \left(\Pi_{r=0}^{n} \frac{1}{(3r+3)(3r+4)}\right) x^{3n+4} \\
&= x\left(1 + \sum_{n=0}^{\infty} \left(\Pi_{r=0}^{n} \frac{1}{(3r+3)(3r+4)}\right) x^{3n+3}\right).
\end{aligned}$$

Power Series Maple Code. It is possible to reproduce the first few terms (below, up to x^{20}) of the power series solutions y_1, y_2 using the computer algebra system `maple`. Here's how:

```
de1:=diff(y1(x),x,x)-x*y1(x)=0; Order:=20;
dsolve({de1,y1(0)=1,D(y1)(0)=0},y1(x),type=series);
de2:=diff(y2(x),x,x)-x*y2(x)=0;
dsolve({de2,y2(0)=0,D(y2)(0)=1},y2(x),type=series);
```

The `maple` global variable Order assigns the number of terms to compute in the series method of `dsolve()`.

The Airy wave functions are defined so that

$$\begin{aligned}\sqrt{3}\,\texttt{AiryAi}(0) &= \texttt{AiryBi}(0) &\approx 0.6149266276,\\ -\sqrt{3}\,\texttt{AiryAi}'(0) &= \texttt{AiryBAi}'(0) &\approx 0.4482883572.\end{aligned}$$

They are *not identical* to y_1, y_2, in particular.

A Taylor Polynomial Method. The first power series solution

$$y(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \frac{1}{1710720}x^{12} + \cdots$$

for Airy's airfoil differential equation $y'' = xy$ can be found without knowing anything about recursion relations or properties of infinite series. Shown here is a Taylor polynomial method which requires only a calculus background. The computation reproduces the answer given by the `maple` code below.

```
de:=diff(y(x),x,x)-x*y(x)=0; Order:=10;
dsolve({de,y(0)=1,D(y)(0)=0},y(x),type=series);
```

The calculus background:

Theorem 4 (Taylor Polynomials)
Let $f(x)$ have $n+1$ continuous derivatives on $a < x < b$ and assume given x_0, $a < x_0 < b$. Then

$$(3) \quad f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + R_n$$

where the remainder R_n has the form

$$R_n = f^{(n+1)}(x_1)\frac{(x - x_0)^{n+1}}{(n+1)!}$$

for some point x_1 between a and b.

The polynomial on the right in (3) is called the **Taylor polynomial** of degree n for $f(x)$ at $x = x_0$. If f is infinitely differentiable, then it has Taylor polynomials of all orders. The **Taylor series** of f is the infinite series obtained formally by letting $n = \infty$ and $R_n = 0$.

For the Airy differential equation problem, $x_0 = 0$. We assume that $y(x)$ is determined by initial conditions $y(0) = 1$, $y'(0) = 0$. The method is a simple one:

> Differentiate the differential equation formally several times, then set $x = x_0$ in all these equations. Resolve from the several equations the values of $y''(x_0)$, $y'''(x_0)$, $y^{iv}(x_0)$, ... and then write out the Taylor polynomial approximation
>
> $$y(x) \approx y(x_0) + y'(x_0)(x - x_0) + y''(x_0)\frac{(x - x_0)^2}{2} + \cdots$$

The successive derivatives of Airy's differential equation are

$$\begin{aligned} y'' &= xy, \\ y''' &= y + xy', \\ y^{iv} &= 2y' + xy'', \\ y^{v} &= 3y'' + xy''', \\ &\vdots \end{aligned}$$

Set $x = x_0 = 0$ in the above equations. Then

$y(0) = 1$	Given.
$y'(0) = 0$	Given.
$y''(0) = xy\vert_{x=0} = 0$	Use Airy's equation $y'' = xy$.
$y'''(0) = (y + xy')\vert_{x=0} = 1$	Use $y''' = y + xy'$.
$y^{iv}(0) = (2y' + xy'')\vert_{x=0} = 0$	Use $y^{iv} = 2y' + xy''$.
$y^{v}(0) = (3y'' + xy''')\vert_{x=0} = 0$	Use $y^{v} = 3y'' + xy'''$.
$y^{vi}(0) = (4y''' + xy^{iv})\vert_{x=0} = 4$	Use $y^{vi} = 4y''' + xy^{iv}$.

Finally, we write out the Taylor polynomial approximation of y:

$$\begin{aligned} y(x) &\approx y(0) + y'(0)x + y''(0)\frac{x^2}{2} + \cdots \\ &= 1 + 0 + 0 + \frac{x^3}{6} + 0 + 0 + \frac{4x^6}{6!} + \cdots \\ &= 1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots \end{aligned}$$

Computer algebra systems can replace the hand work, finding the Taylor polynomial directly.

12.4 Ordinary Points

Developed here is the mathematical theory for second order differential equations and their Taylor series solutions. We assume a differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \quad a(x) \neq 0. \tag{1}$$

Such an equation can always be converted to the **standard form**

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

The conversion from (1) to (2) is made from the formulas

$$p(x) = b(x)/a(x), \quad q(x) = c(x)/a(x).$$

A point $x = x_0$ is called an **ordinary point** of equation (2) provided both $p(x)$ and $q(x)$ have Taylor series expansions valid in an interval $|x - x_0| < R$, $R > 0$. Any point that is not an ordinary point is called a **singular point**. For equation (1), $x = x_0$ is an ordinary point provided $a(x) \neq 0$ at $x = x_0$ and each of $a(x)$, $b(x)$, $c(x)$ has a Taylor series expansion valid in some interval about $x = x_0$.

Theorem 5 (Power series solutions)
Let $a(x)y'' + b(x)y' + c(x)y = 0$, $a(x) \neq 0$, be given and assume that $x = x_0$ is an ordinary point. If the Taylor series of both $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$ are convergent in $|x - x_0| < R$, then the differential equation has two independent Taylor series solutions

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

convergent in $|x - x_0| < R$. Any solution $y(x)$ defined in $|x - x_0| < R$ can be written as $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for a unique set of constants c_2, c_2.

A proof of this result can be found in Birkhoff-Rota [?]. The maximum allowed value of R is the distance from x_0 to the nearest singular point.

Ordinary Point Illustration. We will determine the two independent solutions y_1, y_2 of Theorem 5 for the second order differential equation

$$y'' - 2xy' + y = 0.$$

Let LHS stand for the left side of the differential equation. Assume a trial solution $y = \sum_{n=0}^{\infty} c_n x^n$. Then formulas on pages 701 and 702 imply

$$\begin{aligned} \text{LHS} &= y'' - 2xy' + y \\ &= \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n + \sum_{n=1}^{\infty}(-2)nc_nx^n + \sum_{n=0}^{\infty}c_nx^n \\
&= 2c_2 + c_0 + \sum_{n=1}^{\infty}((n+1)(n+2)c_{n+2} - 2nc_n + c_n)x^n \\
&= 2c_2 + c_0 + \sum_{n=1}^{\infty}((n+1)(n+2)c_{n+2} - (2n-1)c_n)x^n
\end{aligned}
$$

The power series LHS equals the zero power series, which gives rise to the recursion relations $2c_2 + c_0 = 0$, $(n+1)(n+2)c_{n+2} - (2n-1)c_n = 0$, $n \ge 1$, or more succinctly the two-termed second order recursion

$$c_{n+2} = \frac{2n-1}{(n+1)(n+2)}c_n, \quad n \ge 0.$$

Using the formulas on page 706, we obtain the recursion answers

$$
\begin{aligned}
c_{2k+2} &= \left(\Pi_{r=0}^{k}\frac{4r-1}{(2r+1)(2r+2)}\right)c_0, \\
c_{2k+3} &= \left(\Pi_{r=0}^{k}\frac{4r+1}{(2r+2)(2r+3)}\right)c_1.
\end{aligned}
$$

Taking $c_0 = 1$, $c_1 = 0$ gives y_1 and taking $c_0 = 0$, $c_1 = 1$ gives y_2:

$$
\begin{aligned}
y_1(x) &= 1 + \sum_{k=0}^{\infty}\left(\Pi_{r=0}^{k}\frac{4r-1}{(2r+1)(2r+2)}\right)x^{2k+2}, \\
y_2(x) &= x + \sum_{k=0}^{\infty}\left(\Pi_{r=0}^{k}\frac{4r+1}{(2r+2)(2r+3)}\right)x^{2k+3}.
\end{aligned}
$$

These solutions have Wronskian 1 at $x = 0$, hence they are independent and they form a basis for the solution space of the differential equation.

Some `maple` Code. It is possible to directly program the basis y_1, y_2 in `maple`, ready for plotting and computation of solutions to initial value problems. At the same time, we can check the series formulas against the `maple` engine, which is able to solve for the series solutions y_1, y_2 to any order of accuracy.

```
f:=t->(2*t-1)/((t+1)*(t+2)):
c1:=k->product(f(2*r),r=0..k):
c2:=k->product(f(2*r+1),r=0..k):
y1:=(x,N)->1+sum(c1(k)*x^(2*k+2),k=0..N);
y2:=(x,N)->x+sum(c2(k)*x^(2*k+3),k=0..N);
de:=diff(y(x),x,x)-2*x*diff(y(x),x)+y(x)=0: Order:=10:
dsolve({de,y(0)=1,D(y)(0)=0},y(x),type=series); # find y1
'y1'=y1(x,5);
```

```
dsolve({de,y(0)=0,D(y)(0)=1},y(x),type=series); # find y2
'y2'=y2(x,5);
plot(2*y1(x,infinity)+3*y2(x,infinity),x=0..1);
```

The `maple` formulas are

$$y_1(x) = 1 - \frac{1}{2}\,x^2 - \frac{1}{8}\,x^4 - \frac{7}{240}\,x^6 - \frac{11}{1920}\,x^8 + \cdots$$
$$y_2(x) = x + \frac{1}{6}\,x^3 + \frac{1}{24}\,x^5 + \frac{1}{112}\,x^7 + \frac{13}{8064}\,x^9 + \cdots$$

The `maple` approximation of $2y_1 + 3y_2$ to order 20 agrees with the exact solution for the first 8 digits. Often the infinity required for the exact solution can be replaced by an integer $N = 10$ or smaller, to produce exactly the same plot.

12.5 Regular Singular Points

The model differential equation for Frobenius singular point theory is the Cauchy-Euler differential equation

$$ax^2y'' + bxy' + cy = 0. \tag{1}$$

The Frobenius theory treats a **perturbation** of the Cauchy-Euler equation obtained by replacement of the constants a, b, c by Maclaurin power series. Such a **Frobenius differential equation** has the special form

$$x^2a(x)y'' + xb(x)y' + c(x)y = 0$$

where $a(x) \neq 0$, $b(x)$, $c(x)$ have Maclaurin expansions.

The Cauchy-Euler differential equation (1) gives some intuition about the possible kinds of solutions for Frobenius equations. It is known that the Cauchy-Euler differential equation can be transformed to a constant-coefficient differential equation

$$a\frac{d^2z}{dt^2} + (b-a)\frac{dz}{dt} + cz = 0 \tag{2}$$

via the change of variables

$$z(t) = y(e^t), \quad x = e^t.$$

By the constant-coefficient formulas, Theorem 1 in Chapter 6, equation (1) has three kinds of possible solutions, organized by the character of the roots r_1, r_2 of the characteristic equation $ar^2 + (b-a)r + c = 0$ of (2). The three kinds are ($r_1 = \overline{r}_2 = \alpha + i\beta$ in case 3):

Case 1: Discriminant positive	$y = c_1x^{r_1} + c_2x^{r_2}$
Case 2: Discriminant zero	$y = c_1x^{r_1} + c_2x^{r_1}\ln\|x\|$
Case 3: Discriminant negative	$y = c_1x^{\alpha}\cos(\beta\ln\|x\|) + c_2x^{\alpha}\sin(\beta\ln\|x\|)$

The last solution is **singular** at $x = 0$, the location where the leading coefficient in (1) is zero. The second solution is singular at $x = 0$ when $c_2 \neq 0$. The other solutions involve powers x^r; they can be singular solutions at $x = 0$ if $r < 0$.

The Cauchy-Euler conjecture. This conjecture about solutions of Frobenius equations is often made by differential equation rookies:

> Isn't it true that a Frobenius differential equation has a general solution obtained from the general solution of the Cauchy-Euler differential equation
>
> $$x^2a(0)y'' + xb(0)y' + c(0)y = 0$$
>
> by replacement of the the constants c_1, c_2 by Maclaurin power series?

As a tribute to this intuitive conjecturing, we can say in hindsight that the **Cauchy-Euler conjecture** is *almost correct*! Perhaps it is a good way to remember the results of the Frobenius theory, to follow.

Frobenius theory. A **Frobenius differential equation** singular at $x = x_0$ has the form

$$(x - x_0)^2 A(x)y'' + (x - x_0)B(x)y' + C(x)y = 0 \tag{3}$$

where $A(x_0) \neq 0$ and $A(x)$, $B(x)$, $C(x)$ have Taylor series expansions at $x = x_0$ valid in an interval $|x - x_0| < R$, $R > 0$. Such a point $x = x_0$ is called a **regular singular point** of (3). Any other point $x = x_0$ is called an **irregular singular point**.

A Frobenius regular singular point differential equation generalizes the Cauchy-Euler differential equation, because if the Taylor series are constants and the translation $x \to x - x_0$ is made, then the Frobenius equation reduces to a Cauchy-Euler equation.

The **indicial equation** of (3) is the quadratic equation

$$A(x_0)r^2 + (B(x_0) - A(x_0))r + C(x_0) = 0.$$

More precisely, the indicial equation is obtained logically in two steps:

(1) Transform the Cauchy-Euler differential equation

$$(x - x_0)^2 A(x_0)y'' + (x - x_0)B(x_0)y' + C(x_0)y = 0$$

by the change of variables $x - x_0 = e^t$, $z(t) = y(x_0 + e^t)$ to obtain the constant-coefficient differential equation

$$t^2 A(0)\frac{d^2z}{dt^2} + t((B(0) - A(0))\frac{dz}{dt} + C(0)z = 0.$$

(2) Determine the characteristic equation of the constant-coefficient differential equation.

The indicial equation can be used to directly solve Cauchy-Euler differential equations. The roots of the indicial equation plus the constant-coefficient formulas, Theorem 1 in Chapter 6, provide answers which directly transcribe the general solution of the Cauchy-Euler equation.

The Frobenius theory analyzes the Frobenius differential equation only in the case when the roots of the indicial equation are real, which corresponds to the discriminant positive or zero in the discriminant table, page 715.

The cases in which the discriminant is non-negative have their own complications. Expected from the Cauchy-Euler conjecture is a so-called **Frobenius solution**

$$y(x) = (x - x_0)^r \left(c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots\right),$$

in which r is a root of the indicial equation. Two independent Frobenius solutions may or may not exist, therefore the Cauchy-Euler conjecture turns out to be partly correct, but false, in general.

The last case, in which the discriminant of the indicial equation is negative, is not treated here.

Theorem 6 (Frobenius solutions)
Let $x = x_0$ be a regular singular point of the Frobenius equation

$$(x - x_0)^2 A(x) y'' + (x - x_0) B(x) y' + C(x) y = 0. \tag{4}$$

Let the indicial equation $A(x_0)r^2 + (B(x_0) - A(x_0))r + C(x_0) = 0$ have real roots r_1, r_2 with $r_1 \geq r_2$. Then equation (4) always has one Frobenius series solution y_1 of the form

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad c_0 \neq 0.$$

The root r_1 has to be the larger root: the equation can fail for the smaller root r_2.

Equation (4) has a second independent solution y_2 in the following cases.

(a) If $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, then, for some coefficients $\{d_n\}$ with $d_0 \neq 0$,

$$y_2(x) = (x - x_0)^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n.$$

(b) If $r_1 \neq r_2$ and $r_1 - r_2$ is a positive integer, then, for some coefficients $\{d_n\}$ with $d_0 \neq 0$ and either $C = 0$ or $C = 1$,

$$y_2(x) = C y_1(x) \ln|x - x_0| + (x - x_0)^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n.$$

(c) If $r_1 = r_2$, then, for some coefficients $\{d_n\}$ with $d_0 = 0$,

$$y_2(x) = y_1(x) \ln|x - x_0| + (x - x_0)^{r_1} \sum_{n=0}^{\infty} d_n (x - x_0)^n.$$

A proof of the Frobenius theorem can be found in Birkhoff-Rota [**?**].

Independent proofs of the independence of y_1, y_2 plus some details about how to calculate y_1, y_2 appear below in the examples. In part **(b)** of the theorem, the formula compresses two trial solutions into one, but the intent is that they be tried separately, in order $C = 0$, then $C = 1$. Sometimes it is possible to combine the two trials into one complicated computation, but that is not for the faint of heart.

1 Example (Case (a)) Use the Frobenius theory to solve for y_1, y_2 in the differential equation $2x^2y'' + xy' + xy = 0$.

Solution: The indicial equation is $2r^2 + (1-2)r + 0 = 0$ with roots $r_1 = 1/2$, $r_2 = 0$. The roots do not differ by an integer, therefore two independent Frobenius solutions y_1, y_2 exist, according to Theorem 6(a). The answers are

$$\begin{aligned} y_1(x) &= x^{1/2}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \frac{1}{22680}x^4 + \cdots\right), \\ y_2(x) &= x^0\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \frac{1}{2520}x^4 + \cdots\right). \end{aligned}$$

The method. Let r be a variable, to eventually be set to either root $r = r_1$ or $r = r_2$. We expect to compute two solutions $y_1(x, y_1)$, $y_2(x, r_2)$ from

$$y(x, r) = x^r \sum_{n=0}^{\infty} c(n, r)x^n.$$

The symbol $c(n, r)$ plays the role of c_n during the computation, but emphasizes the dependence of the coefficient on the root r.

Independence of y_1, y_2. To test independence, let $c_1y_1(x) + c_2y_2(x) = 0$ for all x. Proving $c_1 = c_2 = 0$ implies y_1, y_2 are independent. Divide the equation $c_1y_1 + c_2y_2 = 0$ by x^{r_2}. The series representations of y_1, y_2 contain a factor x^{r_2} which divides out, leaving two Maclaurin series and a factor of $x^{r_1-r_2}$ on the y_1-series. This factor equals zero at $x = 0$, because $r_1 - r_2 > 0$. Substitution of $x = 0$ shows that $c_2 = 0$. Hence also $c_1 = 0$. The test is complete.

A formula for $c(n, r)$. The method applied is substitution of the series $y(x, r)$ into the differential equation in order to resolve the coefficients. At certain steps, series indexed from zero to infinity are split into the $n = 0$ term plus the rest of the series, in order to match summation ranges. Index changes are used to match powers of x. The details:

$$\begin{aligned}
x^2A(x)y'' &= 2x^2y''(x,r) \\
&= 2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c(n,r)x^{n+r-2} \\
&= 2r(r-1)c(0,r)x^r + \sum_{n=1}^{\infty}2(n+r)(n+r-1)c(n,r)x^{n+r}, \\
xB(x)y' &= xy'(x,r) \\
&= \sum_{n=0}^{\infty}(n+r)c(n,r)x^{n+r} \\
&= rc(0,r)x^r + \sum_{n=1}^{\infty}(n+r)c(n,r)x^{n+r} \\
C(x)y &= xy(x,r) \\
&= \sum_{n=0}^{\infty}c(n,r)x^{n+r+1} \\
&= \sum_{n=1}^{\infty}c(n-1,r)x^{n+r}.
\end{aligned}$$

Recursion. Let $p(r) = 2r(r-1) + r + 0$ be the indicial polynomial. Let LHS stand for the left hand side of the Frobenius differential equation. Add the preceding equations. Then

$$\begin{aligned}
\text{LHS} &= 2x^2y''(x,r) + xy'(x,r) + xy(x,r) \\
&= p(r)c(0,r)x^r + \sum_{n=1}^{\infty}\left(p(n+r)c(n,r) + c(n-1,r)\right)x^{n+r}.
\end{aligned}$$

Because LHS equals the zero series, all coefficients are zero, which implies $p(r) = 0$, $c(0,r) \neq 0$, and the recursion relation

$$p(n+r)c(n,r) + c(n-1,r) = 0, \quad n \geq 1.$$

Solution of the recursion. The recursion answers on page 706 imply for $c_0 = c(0,r) = 1$ the relations

$$\begin{aligned}
c(n+1,r) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{p(k+1+r)}\right) \\
c(n+1,r_1) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{p(k+3/2)}\right) \\
c(n+1,r_2) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{p(k+1)}\right)
\end{aligned}$$

Then $y_1(x) = y(x, r_1)$, $y_2(x) = y(x, r_2)$ imply

$$\begin{aligned}
y_1(x) &= x^{1/2}\left(1+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{(2k+3)(k+1)}\right)x^{n+1}\right) \\
&= x^{1/2}\left(1+\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{n+1}}{(2n+3)!}\,x^{n+1}\right), \\
y_2(x) &= x^{0}\left(1+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{(k+1)(2k+1)}\right)x^{n+1}\right) \\
&= x^{0}\left(1+\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{n}}{(n+1)(2n+1)!}\,x^{n+1}\right).
\end{aligned}$$

Answer checks. It is possible to verify the answers using `maple`, as follows.

```
de:=2*x^2*diff(y(x),x,x)+x*diff(y(x),x)+x*y(x)=0;
Order:=5;dsolve(de,y(x),series);
c:=n->(-1)^(n+1)*product(1/((2*k+3)*(k+1)),k=0..n);
d:=n->(-1)^(n+1)*product(1/((2*k+1)*(k+1)),k=0..n);
1+sum(c(n)*x^(n+1),n=0..6);
1+sum((-1)^(n+1)*2^(n+1)/((2*n+3)!)*x^(n+1),n=0..6);
1+sum(d(n)*x^(n+1),n=0..6);
1+sum((-1)^(n+1)*2^(n)/((n+1)*(2*n+1)!)*x^(n+1),n=0..6);
```

Verified by `maple` is an exact solution formula $y(x) = c_1\cos(\sqrt{2x}) + c_2\sin(\sqrt{2x})$ in terms of elementary functions. The code details:

```
de:=2*x^2*diff(y(x),x,x)+x*diff(y(x),x)+x*y(x)=0;
dsolve(de,y(x));
```

2 Example (Case (b)) Use the Frobenius theory to solve for y_1, y_2 in the differential equation $x^2y'' + x(3+x)y' - 3y = 0$.

Solution: The indicial equation is $r^2 + (3-1)r - 3 = 0$ with roots $r_1 = 1$, $r_2 = -3$. The roots differ by an integer, therefore one Frobenius solution y_1 exists and the second independent solution y_2 must be computed according to Theorem 6(b). The answers are

$$\begin{aligned}
y_1(x) &= x\left(1-\frac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1680}x^4+\cdots\right), \\
y_2(x) &= x^{-3}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3\right).
\end{aligned}$$

Let r denote either root r_1 or r_2. We expect to compute solutions y_1, y_2 by the following scheme.

$$\begin{aligned}
y(x,r) &= x^r\sum_{n=0}^{\infty}c(n,r)x^n, \\
y_1(x) &= y(x,r_1), \\
y_2(x) &= Cy_1(x)\ln(x) + x^{r_2}\sum_{n=0}^{\infty}d_nx^n.
\end{aligned}$$

The constant C is either zero or one, but the value cannot be decided until the end of the computation. Likewise, $d_0 \neq 0$ is known, but little else about the sequence $\{d_n\}$ is known.

Finding a formula for $c(n, r)$. The method substitutes the series $y(x, r)$ into the differential equation and then solves for the undetermined coefficients. The details:

$$\begin{aligned}
x^2A(x)y'' &= x^2y''(x,r) \\
&= x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c(n,r)x^{n+r-2} \\
&= r(r-1)c(0,r)x^r + \sum_{n=1}^{\infty}(n+r)(n+r-1)c(n,r)x^{n+r}, \\
xB(x)y' &= (3+x)xy'(x,r) \\
&= (3+x)x\sum_{n=0}^{\infty}(n+r)c(n,r)x^{n+r-1} \\
&= \sum_{n=0}^{\infty}3(n+r)c(n,r)x^{n+r} + \sum_{n=0}^{\infty}(n+r)c(n,r)x^{n+r+1} \\
&= 3rc(0,r)x^r + \sum_{n=1}^{\infty}3(n+r)c(n,r)x^{n+r} \\
&\quad + \sum_{n=1}^{\infty}(n+r-1)c(n-1,r)x^{n+r}, \\
C(x)y &= -3y(x,r) \\
&= -3c(0,r)x^r + \sum_{n=1}^{\infty}-3c(n,r)x^{n+r}.
\end{aligned}$$

Finding the recursions. Let $p(r) = r(r-1)+3r-3$ be the indicial polynomial. Let LHS denote the left hand side of $x^2y''+x(3+x)y'-3y = 0$. Add the three equations above. Then

$$\begin{aligned}
\text{LHS} &= x^2y''(x,r) + (3+x)xy'(x,r) - 3y(x,r) \\
&= p(r)c(0,r)x^r + \sum_{n=1}^{\infty}\left(p(n+r)c(n,r) + (n+r-1)c(n-1,r)\right)x^{n+r}.
\end{aligned}$$

Symbol LHS equals the zero series, therefore all the coefficients are zero. Given $c(0, r) \neq 0$, then $p(r) = 0$ and we have the recursion relation

$$p(n+r)c(n,r) + (n+r-1)c(n-1,r) = 0, \quad n \geq 1.$$

Solving the recursion. Using $c(0, r) = 1$ and the recursion answers on page 706 gives

$$\begin{aligned}
c(n+1,r) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{k+r}{p(k+1+r)}\right) \\
c(n+1,1) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{k+1}{(k+1)(k+5)}\right) \\
&= (-1)^{n+1}\frac{24}{(n+5)!}
\end{aligned}$$

Therefore, the first few coefficients $c_n = c(n, 1)$ of y_1 are given by

$$c_0 = 1, \quad c_1 = \frac{-1}{5}, \quad c_2 = \frac{1}{30}, \quad c_3 = \frac{-1}{210}, \quad c_4 = \frac{1}{1680}.$$

This agrees with the reported solution y_1, whose general definition is

$$y_1(x) = 1 + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{24}{(n+5)!}\,x^{n+1}.$$

Finding the second solution y_2. Let's assume that $C = 0$ in the trial solution y_2. Let $d_n = c(n, r_2)$. Then the preceding formulas give the recursion relations

$$p(r_2)d_0 = 0, \quad p(n+r_2)d_n + (n+r_2-1)d_{n-1} = 0, \quad n \ge 1.$$

We require $r_2 = -3$ and $d_0 \ne 0$. The recursions reduce to

$$p(n-3)d_n + (n-4)d_{n-1} = 0, \quad n \ge 1.$$

The solution for $0 \le n \le 3$ is found from $d_n = -\dfrac{n-4}{p(n-3)}d_{n-1}$:

$$d_0 \ne 0, \quad d_1 = -d_0, \quad d_2 = \frac{1}{2}d_0, \quad d_3 = -\frac{1}{6}d_0.$$

There is no condition at $n = 4$, leaving d_4 arbitrary. This gives the recursion

$$p(n+2)d_{n+5} + (n+1)d_{n+4} = 0, \quad n \ge 0.$$

The solution of this recursion is

$$\begin{aligned} d_{n+5} &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{k+1}{p(k+2)}\right)d_4 \\ &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{k+1}{(k+1)(k+5)}\right)d_4 \\ &= (-1)^{n+1}\frac{24}{(n+5)!}d_4. \end{aligned}$$

Momentarily let $d_4 = 1$. Then

$$d_4 = 1, \quad d_5 = -\frac{1}{5}, \quad d_6 = \frac{1}{30}, \quad d_7 = -\frac{1}{210},$$

and then the series terms for $n = 4$ and higher equal

$$x^{-3}\left(x^4 - \frac{1}{5}x^5 + \frac{1}{30}x^6 - \frac{1}{210}x^7 + \cdots\right) = y_1(x).$$

This implies

$$\begin{aligned} y_2(x) &= x^{-3}\left(d_0 + d_1x + d_2x^2 + d_3x^3\right) + d_4y_1(x) \\ &= x^{-3}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)d_0 + d_4y_1(x). \end{aligned}$$

By superposition, y_1 can be dropped from the formula for y_2. The conclusion for case $C = 0$ is

$$y_2(x) = x^{-3}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right).$$

False path for $C = 1$. We take $C = 1$ and repeat the derivation of y_2, just to see why this path leads to no solution with a $\ln(x)$-term. We have a 50%

chance in Frobenius series problems of taking the wrong path to the solution. We will see details for success and also the signal for failure.

Decompose $y_2 = A + B$ where $A = y_1(x)\ln(x)$ and $B = x^{r_2}\sum_{n=1}^{\infty} d_n x^n$. Let $L(y) = x^2y'' + x(3+x)y' - 3y$ denote the left hand side of the Frobenius differential equation. Then $L(y_2) = 0$ becomes $L(B) = -L(A)$.

Compute $L(B)$. The substitution of B into the differential equation to obtain LHS has been done above. Let $d_n = c(n, r_2)$, $r_2 = -3$. The equation $p(r_2) = 0$ eliminates the extra term $p(r_2)c(0, r_2)x^{r_2}$. Split the summation into $1 \le n \le 4$ and $5 \le n < \infty$. Change index $n = m + 4$ to obtain:

$$\begin{aligned} L(B) &= \sum_{n=1}^{\infty} \left(p(n+r_2)c(n,r_2) + (n+r_2-1)c(n-1,r_2)\right) x^{n+r_2} \\ &= \sum_{n=1}^{3} \left(p(n-3)d_n + (n-4)d_{n-1}\right) x^{n-3} + (p(1)d_4 + (0)d_3)x \\ &\quad + \sum_{m=1}^{\infty} \left(p(m+1)d_{m+4} + (m)d_{m+3}\right) x^{m+1}. \end{aligned}$$

Compute $L(A)$. Use $L(y_1) = 0$ in the third step and $r_1 = 1$ in the last step, below.

$$\begin{aligned} L(A) &= x^2(y_1''\ln(x) + 2x^{-1}y_1' - x^{-2}y_1) \\ &\quad +(3+x)x(y_1'\ln(x) + x^{-1}y_1) - 3y_1\ln(x) \\ &= L(y_1)\ln(x) + (2+x)y_1 + 2xy_1' \\ &= (2+x)y_1 + 2xy_1' \\ &= \sum_{n=0}^{\infty} 2c_n x^{n+r_1} + \sum_{n=1}^{\infty} c_{n-1}x^{n+r_1} + \sum_{n=0}^{\infty} 2(n+r_1)c_n x^{n+r_1} \\ &= 4c_0x + \sum_{n=1}^{\infty} ((2n+4)c_n + c_{n-1})x^{n+1}. \end{aligned}$$

Find $\{d_n\}$. The equation $L(B) = -L(A)$ produces recursion relations by matching corresponding powers of x on each side of the equality. We are given $d_0 \ne 0$. For $1 \le n \le 3$, the left side matches zero coefficients on the right side, therefore as we saw in the case $C = 0$,

$$d_0 \ne 0, \quad d_1 = -d_0, \quad d_2 = \frac{1}{2}d_0, \quad d_3 = -\frac{1}{6}d_0.$$

The term for $n = 4$ on the left is $(p(1)d_4 + (0)d_3)x$, which is always zero, regardless of the values of d_3, d_4. On the other hand, there is the **nonzero term** $4c_0x$ on the right. We can never match terms, therefore there is **no solution** with $C = 1$. This is the only *signal for failure*.

Independence of y_1**,** y_2. Two functions y_1, y_2 are called independent provided $c_1y_1(x) + c_2y_2(x) = 0$ for all x implies $c_1 = c_2 = 0$. For the given solutions, we test independence by solving for c_1, c_2 in the equation

$$c_1x\left(1 - \frac{1}{5}x + \frac{1}{30}x^2 - \frac{1}{210}x^3 + \cdots\right) + c_2x^{-3}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right) = 0.$$

Divide the equation by x^{r_2}, then set $x = 0$. We get $c_2 = 0$. Substitute $c_2 = 0$ in the above equation. Divide by x^{r_1}, then set $x = 0$ to obtain $c_1 = 0$. Therefore, $c_1 = c_2 = 0$ and the test is complete.

Answer checks. The simplest check uses `maple` as follows. It is interesting that both y_1 and y_2 are expressible in terms of elementary functions, seen by executing the code below, and detected as a matter of course by `maple dsolve()`.

```
de:=x^2*diff(y(x),x,x)+x*(3+x)*diff(y(x),x)+(-3)*y(x)=0;
Order:=5;dsolve({de},y(x),type=series);
c:=n->(-1)^(n+1)*product((k+1)/((k+5)*(k+1)),k=0..n);
y1:=x+sum(c(n)*x^(n+2),n=0..5);
x+sum(c(n)*x^(n+2),n=0..infinity);
y2:=x->x^(-3)*( 1-x + x^2/2 -(1/6)*x^3);
simplify(subs(y(x)=y2(x),de));
dsolve(de,y(x));
```

3 Example (Case (c)) Use the Frobenius theory to solve for y_1, y_2 in the differential equation $x^2y'' + x(3+x)y' + y = 0$.

Solution: The indicial equation is $r^2 + (3-1)r + 1 = 0$ with roots $r_1 = -1$, $r_2 = -1$. The roots are equal, therefore one Frobenius solution y_1 exists and the second independent solution y_2 must be computed according to Theorem 6. The answers:

$$\begin{aligned} y_1(x) &= x^{-1}(1+x), \\ y_2(x) &= x^{-1}\left(-3x - \frac{1}{4}x^2 + \frac{1}{36}x^3 - \frac{1}{288}x^4 + \frac{1}{2400}x^5 + \cdots\right) \end{aligned}$$

Trial solution formulas for y_1, y_2. Based upon the proof details of the Frobenius theorem, we expect to compute the solutions as follows.

$$\begin{aligned} y(x,r) &= x^r \sum_{n=0}^{\infty} c(n,r)x^n, \\ y_1(x) &= y(x,r_1), \\ y_2(x) &= \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=r_1} \\ &= \left.\left(y(x,r)\ln(x) + x^r \sum_{n=0}^{\infty} \frac{\partial c(n,r)}{\partial r}x^n\right)\right|_{r=r_1} \\ &= y(x,r_1)\ln(x) + x^{r_1}\sum_{n=1}^{\infty} d_n x^n \end{aligned}$$

for some constants d_1, d_2, d_3, ... In some applications, it seems easier to use the partial derivative formula, in others, the final expression in symbols $\{d_n\}$ is more tractable. Finally, we might reject both methods in favor of the reduction of order formula for y_2.

Independence of y_1, y_2. To test independence, let $c_1y_1(x) + c_2y_2(x) = 0$ for all x. Proving $c_1 = c_2 = 0$ implies y_1, y_2 are independent. Divide the equation $c_1y_1 + c_2y_2 = 0$ by x^{r_1}. The series representations of y_1, y_2 contain a factor x^{r_1} which divides out, leaving two Maclaurin series and a $\ln(x)$-term. Then $\ln(0) = -\infty$, $c(0,r_1) \neq 0$ and the finiteness of the series shows that $c_2 = 0$. Hence also $c_1 = 0$. This completes the test.

Finding a formula for $c(n,r)$. The method is to substitute the series $y(x,r)$ into the differential equation and then resolve the coefficients. The details:

$$\begin{aligned}
x^2A(x)y'' &= x^2y''(x,r) \\
&= x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c(n,r)x^{n+r-2} \\
&= r(r-1)c(0,r)x^r + \sum_{n=1}^{\infty}(n+r)(n+r-1)c(n,r)x^{n+r}, \\
xB(x)y' &= (3+x)xy'(x,r) \\
&= (3+x)x\sum_{n=0}^{\infty}(n+r)c(n,r)x^{n+r-1} \\
&= \sum_{n=0}^{\infty}3(n+r)c(n,r)x^{n+r} + \sum_{n=0}^{\infty}(n+r)c(n,r)x^{n+r+1} \\
&= 3rc(0,r)x^r + \sum_{n=1}^{\infty}3(n+r)c(n,r)x^{n+r} \\
&\quad + \sum_{n=1}^{\infty}(n+r-1)c(n-1,r)x^{n+r}, \\
C(x)y &= y(x,r) \\
&= c(0,r)x^r + \sum_{n=1}^{\infty}c(n,r)x^{n+r}.
\end{aligned}$$

Finding the recursions. Let $p(r) = r(r-1)+3r+1$ be the indicial polynomial. Let LHS stand for the left hand side of the Frobenius differential equation. Add the above equations. Then

$$\begin{aligned}
\text{LHS} &= x^2y''(x,r) + (3+x)xy'(x,r) + y(x,r) \\
&= p(r)c(0,r)x^r + \sum_{n=1}^{\infty}\left(p(n+r)c(n,r) + (n+r-1)c(n-1,r)\right)x^{n+r}.
\end{aligned}$$

Because LHS equals the zero series, all coefficients are zero, which implies $p(r) = 0$ for $c(0,r) \neq 0$, plus the recursion relation

$$p(n+r)c(n,r) + (n+r-1)c(n-1,r) = 0, \quad n \geq 1.$$

Solving the recursions. Using the recursion answers on page 706 gives

$$\begin{aligned}
c(n+1,r) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{k+r}{p(k+1+r)}\right)c(0,r) \\
c(n+1,-1) &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{k-1}{(k+1)^2}\right)c(0,r).
\end{aligned}$$

Therefore, $c(0,-1) \neq 0$, $c(1,-1) = c(0,-1)$, $c(n+1,-1) = 0$ for $n \geq 1$.

A formula for y_1. Choose $c(0,-1) = 1$. Then the formula for $y(x,r)$ and the requirement $y_1(x) = y(x,r_1)$ gives

$$y_1(x) = x^{-1}(1+x).$$

A formula for y_2. Of the various expressions for the solution, we choose

$$y_2(x) = y_1(x)\ln(x) + x^{r_1}\sum_{n=1}^{\infty}d_nx^n.$$

Let us put the trial solution y_2 into the differential equation left hand side $L(y) = x^2y'' + x(3+x)y' + y$ in order to determine the undetermined coefficients $\{d_n\}$. Arrange the computation as $y_2 = A + B$ where $A = y_1(x)\ln(x)$ and $B = x^{r_1}\sum_{n=1}^{\infty} d_n x^n$. Then $L(y_2) = L(A) + L(B) = 0$, or $L(B) = -L(A)$. The work has already been done for series B, because of the work with $y(x,r)$ and LHS. We define $d_0 = c(0,r_1) = 0$, $d_n = c(n,r_1)$ for $n \ge 1$. Then

$$L(B) = 0 + \sum_{n=1}^{\infty} \left(p(n+r)d_n + (n+r-1)d_{n-1}\right) x^{n+r_1}.$$

A direct computation, tedious and routine, gives

$$L(A) = 3 + x.$$

Comparing terms in the equation $L(B) = -L(A)$ results in the recursion relations

$$d_1 = -3, \quad d_2 = -\frac{1}{4}, \quad d_{n+1} = -\frac{n-1}{(n+1)^2}\, d_n \quad (n \ge 2).$$

Solving for the first few terms duplicates the coefficients reported earlier:

$$d_1 = -3, \quad d_2 = -\frac{1}{4}, \quad d_3 = \frac{1}{36}, \quad d_4 = \frac{-1}{288}, \quad d_5 = \frac{1}{2400}.$$

A complete formula:

$$\begin{aligned}
y_2(x) &= x^{-1}\left((1+x)\ln(x) - 3x - \frac{1}{4}x^2 + \frac{1}{4}\sum_{n=2}^{\infty}(-1)^n\left(\Pi_{k=2}^{n}\frac{k-1}{p(k)}\right)x^{n+1}\right) \\
&= x^{-1}\left((1+x)\ln(x) - 3x - \frac{1}{4}x^2 + \sum_{n=2}^{\infty}(-1)^n\frac{(n-1)!}{((n+1)!)^2}\,x^{n+1}\right) \\
&= x^{-1}\left((1+x)\ln(x) - 3x - \frac{1}{4}x^2 + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n+1)}\,\frac{x^{n+1}}{(n+1)!}\right).
\end{aligned}$$

Answer check. The solutions displayed here can be checked in `maple` as follows.

```
de:=x^2*diff(y(x),x,x)+x*(3+x)*diff(y(x),x)+y(x);
y1:=((1+x)/x)*ln(x);
eqA:=simplify(subs(y(x)=y1,de));
dsolve(de=0,y(x),series);
d:=n->(-1)^(n-1)/((n-1)*n*(n!));
y2:=x^(-1)*((1+x)*ln(x)-3*x-x^2/4+sum(d(n+1)*x^(n+1),n=2..6));
```

12.6 Bessel Functions

The work of Friedrich W. Bessel (1784-1846) on planetary orbits led to his 1824 derivation of the equation known in this century as the **Bessel differential equation** or order p:

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

This equation appears in a 1733 work on hanging cables, by Daniel Bernoulli (1700-1782). A particular solution y is called a **Bessel function**. While any real or complex value of p may be considered, we restrict the case here to $p \geq 0$ an integer.

Frobenius theory applies directly to Bessel's equation, which has a regular singular point at $x = 0$. The indicial equation is $r^2 - p^2 = 0$ with roots $r_1 = p$ and $r_2 = -p$. The assumptions imply that cases (b) and (c) of the Frobenius theorem apply: either $r_1 - r_2 =$ positive integer [case (b)] or else $r_1 = r_2 = 0$ and $p = 0$ [case (c)]. In both cases there is a Frobenius series solution for the larger root. This solution is referenced as $J_p(x)$ in the literature, and called a **Bessel function of nonnegative integral order** p. The formulas most often used appear below.

$$\begin{aligned}
J_p(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{p+2n}}{n!(p+n)!}, \\
J_0(x) &= 1 - (x/2)^2 + \frac{(x/2)^4}{4^2} - \frac{(x/2)^6}{6^2} + \cdots \\
J_1(x) &= \frac{x}{2} - \frac{(x/2)^3}{(1)(2)} + \frac{(x/2)^5}{(2)(6)} - \frac{(x/2)^7}{(6)(24)} + \cdots
\end{aligned}$$

The derivation of the formula for J_p is obtained by substitution of the trial solution $y = x^r \sum_{n=0}^{\infty} c_n x^n$ into Bessel's equation. Let $Q(r) = r(r-1) - p^2$ be the indicial polynomial. The result is

$$\sum_{n=0}^{\infty} Q(n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+p+2} = 0.$$

Matching terms on the left to the zero coefficients on the right gives the recursion relations

$$Q(r)c_0 = 0, \quad Q(r+1)c_1 = 0, \quad Q(n+r)c_n + c_{n-2} = 0, \quad n \geq 2.$$

To resolve the relations, let $r = p$ (the larger root), $c_0 = 1$, $c_1 = 0$ (because $Q(p+1) \neq 0$), and

$$c_{n+2} = \frac{-1}{Q(n+2+p)} c_n.$$

This is a two-termed second order recursion which can be solved with formulas already developed to give

$$\begin{aligned}
c_{2n+2} &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{(2k+2+p)^2-p^2}\right)c_0 \\
&= (-1)^{n+1}\Pi_{k=0}^{n}\frac{1}{4(k+1)(k+1+p)} \\
&= \frac{(-1)^{n+1}}{4^{n+1}}\frac{1}{(n+1)!}\frac{p!}{(n+1+p)!} \\
&= (2^p p!)\frac{(-1)^{n+1}}{2^{2n+2+p}}\frac{1}{(n+1)!}\frac{1}{(n+1+p)!} \\
c_{2n+3} &= (-1)^{n+1}\left(\Pi_{k=0}^{n}\frac{1}{(2k+3+p)^2-p^2}\right)c_1 \\
&= 0.
\end{aligned}$$

The common factor $(2^p p!)x^p$ can be factored out from each term except the first, which is c_0x^p or x^p. Dividing the answer so obtained by $(2^p p!)$ gives the series reported for J_p.

Properties of Bessel Functions. Sine and cosine identities from trigonometry have direct analogs for Bessel functions. We would like to say that $\cos(x) \leftrightarrow J_0(x)$, and $\sin(x) \leftrightarrow J_1(x)$, but that is not exactly correct. There are asymptotic formulas

$$\begin{aligned}
J_0(x) &\approx \sqrt{\tfrac{2}{\pi x}}\cos\left(x-\tfrac{\pi}{4}\right), \\
J_1(x) &\approx \sqrt{\tfrac{2}{\pi x}}\sin\left(x-\tfrac{\pi}{4}\right).
\end{aligned}$$

See the reference by G.N. Watson [**?**] for details about these asymptotic formulas. At a basic level, based upon the series expressions for J_0 and J_1, the following identities can be easily checked.

Bessel Functions			Trig Functions		
$J_0(0)$	$=$	1	$\cos(0)$	$=$	1
$J_0'(0)$	$=$	0	$(\cos(x))'\big\vert_{x=0}$	$=$	0
$J_1(0)$	$=$	0	$\sin(0)$	$=$	0
$J_1'(0)$	$=$	$1/2$	$(\sin(x))'\big\vert_{x=0}$	$=$	1
$J_0(-x)$	$=$	$J_0(x)$	$\cos(-x)$	$=$	$\cos(x)$
$J_1(-x)$	$=$	$-J_1(x)$	$\sin(-x)$	$=$	$-\sin(x)$

Some deeper relations exist, obtained by series expansion of both sides of the identities. Suggestions for the derivations are in the exercises. The basic reference [?] can be consulted to find complete details.

$$
\begin{aligned}
J_0'(x) &= -J_1(x) \\
J_1'(x) &= J_0(x) - \frac{1}{x} J_1(x) \\
(x^p J_p(x))' &= x^p J_{p-1}(x), \quad p \geq 1, \\
\left(x^{-p} J_p(x)\right)' &= -x^{-p} J_{p+1}(x), \quad p \geq 0, \\
J_{p+1} &= \frac{2p}{x} J_{p+1}(x) - J_{p-1}(x), \quad p \geq 1, \\
J_{p+1}(x) &= -2J_p'(x) + J_{p-1}(x), \quad p \geq 1.
\end{aligned}
$$

The Zeros of Bessel Functions. It is a consequence of the second order differential equation for Bessel functions that these functions have infinitely many zeros on the positive x-axis. As seen from asymptotic expansions, the zeros of J_0 satisfy $x - \pi/4 \approx (2n-1)\pi/2$ and the zeros of J_1 satisfy $x - \pi/4 \approx n\pi$. These approximations are already accurate to one decimal digit for the first five zeros, as seen from the following table.

The positive zeros of J_0 and J_1				
n	$J_0(x)$	$J_1(x)$	$\left(\frac{2n-1}{2} + \frac{1}{4}\right)\pi$	$n\pi + \frac{\pi}{4}$
1	2.40482556	3.83170597	2.35619449	3.92699082
2	5.52007811	7.01558667	5.49778714	7.06858347
3	8.65372791	10.17346813	8.63937980	10.21017613
4	11.79153444	13.32369194	11.78097245	13.35176878
5	14.93091771	16.47063005	14.92256511	16.49336143

The values are conveniently obtained by the following `maple` code.

```
seq(evalf(BesselJZeros(0,n)),n=1..5);
seq(evalf(BesselJZeros(1,n)),n=1..5);
seq(evalf((2*n-1)*Pi/2+Pi/4),n=1..5);
seq(evalf((n)*Pi+Pi/4),n=1..5);
```

The Sturm theory of oscillations of second order differential equations provides the theory which shows that Bessel functions oscillate on the positive x-axis. Part of that theory translates to the following theorem about the interlaced zero property. Trigonometric graphs verify the interlaced zero property for sine and cosine. The theorem for $p = 0$ says that the zeros of $J_0(x) \leftrightarrow \cos(x)$ and $J_1(x) \leftrightarrow \sin(x)$ are interlaced.

Theorem 7 (Interlaced Zeros)
Between pairs of zeros of J_p there is a zero of J_{p+1} and between zeros of J_{p+1} there is a zero of J_p. In short, the zeros of J_p and J_{p+1} are interlaced.

Exercises 12.6

Values of J_0 and J_1. Compute using the series representations, identities and suggestions the decimal values of the following functions. Use a computer algebra system to check the answers.

1. $J_0(1)$

2. $J_1(1)$

3. $J_0(1/2)$

4. $J_1(1/2)$

Proofs of properties. Prove the following relations, using the suggestions supplied.

5. $J_0'(x) = -J_1(x)$

6. $J_1'(x) = J_0(x) - \dfrac{1}{x} J_1(x)$

7. $(x^p J_p(x))' = x^p J_{p-1}(x)$, $p \geq 1$

8. $\left(x^{-p} J_p(x)\right)' = -x^{-p} J_{p+1}(x)$, $p \geq 0$

9. $J_{p+1} = \dfrac{2p}{x} J_{p+1}(x) - J_{p-1}(x)$, $p \geq 1$

10. $J_{p+1}(x) = -2J_p'(x) + J_{p-1}(x)$, $p \geq 1$

12.7 Legendre Polynomials

The differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is called the **Legendre differential equation** of order n, after the French mathematician Adrien Marie Legendre (1752-1833), because of his work on gravitation.[1] The value of n is a nonnegative integer. For each n, the corresponding Legendre equation is known to have a polynomial solution $P_n(x)$ of degree n, called the nth **Legendre polynomial**. The first few of these are recorded below.

$P_0(x) = 1$	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$
$P_1(x) = x$	
$P_2(x) = \frac{3}{2}x^2 - 1/2$	$P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x,$
$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$	$P_6(x) = \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}.$

The general formula for $P_n(x)$ is obtained by using ordinary point theory on Legendre's differential equation. The polynomial is normalized to satisfy $P_n(1) = 1$. The **Legendre polynomial of order** n:

(1) $$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{N} \frac{(-1)^k(2n-2k)!}{k!(n-2k)!(n-k)!}\, x^{n-2k},$$

where $N = \begin{cases} n/2 & n \text{ even,} \\ (n-1)/2 & n \text{ odd.} \end{cases}$.

There are alternative formulas available from which to compute P_n. The most famous one is named after the French economist and mathematician Olinde Rodrigues (1794-1851), known in the literature as **Rodrigues' formula**:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

Equally famous is **Bonnet's recursion**

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x),$$

which was used to produce the table of Legendre polynomials above. The derivation of Bonnet's formula appears in the exercises.

[1] Legendre is recognized more often for his 40 years of work on elliptic integrals.

Properties of Legendre Polynomials. The main relations known for Legendre polynomials P_n are recorded here.

$$
\begin{aligned}
P_n(1) &= 1 \\
P_n(-1) &= (-1)^n \\
P_{2n+1}(0) &= 0 \\
P'_{2n}(0) &= 0 \\
P_n(-x) &= (-1)^n P_n(x) \\
(n+1)P_{n+1}(x) + nP_{n-1}(x) &= (2n+1)xP_n(x) \\
P'_{n+1}(x) - P'_{n-1}(x) &= (2n+1)P_n(x) \\
P'_{n+1}(x) - xP'_n(x) &= (n+1)P_n(x) \\
(1-2xt+t^2)^-1/2 &= \sum_{n=0}^{\infty} P_n(x)t^n \\
\int_{-1}^{1} |P_n(x)|^2 dx &= \frac{2}{2n+1} \\
\int_{-1}^{1} P_n(x)P_m(x)dx &= 0 \quad (n \neq m)
\end{aligned}
$$

Gaussian Quadrature. A high-speed low overhead numerical procedure, called **Gaussian quadrature** in the literature, is defined in terms of the zeros $\{x_k\}_{k=1}^n$ of $P_n(x) = 0$ in $-1 < x < 1$ and certain constants $\{a_k\}_{k=1}^n$ by the approximation formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=1}^{n} a_k f(x_k).$$

The approximation is exact when f is a polynomial of degree less than $2n$. This fact is enough to evaluate the sequence of numbers $\{a_k\}_{k=1}^n$, because we can replace f by the basis functions $1, x, \ldots, x^{n-1}$ to get an $n \times n$ system for the variables $a_1, \ldots, a_n$. The last critical element: the sequence $\{x_k\}_{k=1}^n$ is the set of n distinct roots of $P_n(x) = 0$ in $-1 < x < 1$. Here we need some theory, that says that these roots number n and are all real.

Theorem 8 (Roots of Legendre Polynomials)
The Legendre polynomial P_n has exactly n distinct real roots $x_1, \ldots, x_n$ located in the interval $-1 < x < 1$.

The importance of the Gaussian quadrature formula lies in the ability to make a table of values that generates the approximation, except for the evaluations $f(x_k)$. This makes Gaussian quadrature a very high speed method, because it is based upon function evaluation and a dot product for a fixed number of vector entries. Vector parallel computers are known for their ability to perform these operations at high speed.

A question: *How is Gaussian quadrature different than the rectangular rule*? They are similar methods in the arithmetic requirements of function evaluation and dot product. The rectangular rule has less accuracy than Gaussian quadrature.

Gaussian quadrature can be compared with Simpson's rule. For $n = 3$, which uses three function evaluations, Gaussian quadrature becomes

$$\int_{-1}^{1} f(x)dx \approx \frac{5f(\sqrt{.6}) + 8f(0) + 5f(-\sqrt{.6})}{9},$$

whereas Simpson's rule with one interval is

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1).$$

The reader is invited to compare the two approximations using polynomials f of degree higher than 4, or perhaps a smooth positive function f on $-1 < x < 1$, say $f(x) = \cos(x)$.

Table generation. The pairs (x_j, a_j), $1 \le j \le n$, required for the right side of the Gaussian quadrature formula, can be generated just once for a given n by the following `maple` procedure.

```
GaussQuadPairs:=proc(n)
 local a,x,xx,ans,p,eqs;
 xx:=fsolve(orthopoly[P](n,x)=0,x);
 x:=array(1..n,[xx]);
 eqs:=seq(sum(a[j]*x[j]^k,j=1..n)=int(t^k,t=-1..1),
  k=0..n-1);
 ans:=solve({eqs},{seq(a[j],j=1..n)});
 assign(ans);
 p:=[seq([x[j],a[j]],j=1..n)];
 end proc;
```

For simple applications, the `maple` code above can be attached to the application to generate the table on-the-fly. To generate tables, such as the one below, run the procedure for a given n, e.g., to generate the table for $n = 5$, load or type the procedure given above, then use the command `GaussQuadPairs(5);`.

Table 1. Gaussian Quadrature Pairs for $n = 5$

j	x_j	a_j
1	−0.9061798459	0.2369268851
2	−0.5384693101	0.4786286705
3	0.0000000000	0.5688888887
4	0.5384693101	0.4786286705
5	0.9061798459	0.2369268851

Derivation of the Legendre Polynomial Formula. Let us start with the differential equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

where λ is a real constant. It will be shown that the differential equation has a polynomial solution if and only if $\lambda = n(n+1)$ for some nonnegative integer n, in which case the polynomial solution P_n is given by equation (1).

Proof: The trial solution is a Maclaurin series ($x = 0$ is an *ordinary point*)

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$\begin{aligned}
(1-x^2)y'' &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{n=2}^{\infty} n(n-1)c_n x^n, \\
-2xy' &= \sum_{n=1}^{\infty} -2nc_n x^n, \\
\lambda y &= \sum_{n=0}^{\infty} \lambda c_n x^n.
\end{aligned}$$

Let $L(y) = (1-x^2)y'' - 2xy' + \lambda y$, then, adding the above equations,

$$\begin{aligned}
L(y) &= (1-x^2)y'' - 2xy' + \lambda y \\
&= (2c_2 + \lambda c_0) + (6c_3 - 2c_1 + \lambda c_1)x \\
&\quad + \sum_{n=2}^{\infty}((n+2)(n+1)c_{n+2} + (-n(n-1) - 2n + \lambda)c_n)x^n.
\end{aligned}$$

The requirement $L(y) = 0$ makes the left side coefficients equal the coefficients of the zero series, giving the relations

$$\begin{aligned}
&2c_2 + \lambda c_0 = 0, \\
&6c_3 - 2c_1 + \lambda c_1 = 0, \\
&(n+2)(n+1)c_{n+2} + (-n(n-1) - 2n + \lambda)c_n = 0 \quad (n \ge 2).
\end{aligned}$$

These compress to a single two-termed second order recursion

$$c_{n+2} = \frac{n^2 + n - \lambda}{(n+2)(n+1)} c_n = 0, \quad (n \ge 0),$$

whose solution is

$$\begin{aligned}
c_{2n+2} &= \left(\Pi_{k=0}^{n} \frac{2k(2k+1) - \lambda}{(2k+1)(2k+2)}\right) c_0, \\
c_{2n+3} &= \left(\Pi_{k=0}^{n} \frac{(2k+1)(2k+2) - \lambda}{(2k+2)(2k+3)}\right) c_1.
\end{aligned}$$

Let y_1 be the series solution using $c_0 = 1$, $c_1 = 0$ and let y_2 be the series solution using $c_0 = 0$, $c_1 = 1$.

From the product form of the coefficients, it is apparent that $\lambda = n(n+1)$ for some integer $n \geq 0$ implies one of the two series solutions y_1, y_2 is a polynomial, due to either $c_{2j+2} = 0$ or $c_{2j+3} = 0$ for $j \geq n$.

If some solution y is a polynomial, then $y = d_1 y_1 + d_2 y_2$ and $y^r \equiv 0$ for some integer r. Then d_1, d_2 satisfy the 2×2 linear system

$$\begin{pmatrix} y_1^r(0) & y_2^r(0) \\ y_1^{r+1}(0) & y_2^{r+1}(0) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If λ is not a product $n(n+1)$ for some integer n, then the determinant of coefficients does not equal zero, because of the formula for the coefficients of a Maclaurin series. Cramer's rule applies and $d_1 = d_2 = 0$. Therefore, $y \neq 0$ implies $\lambda = n(n+1)$ for some integer n.

It remains to simplify the coefficients in the polynomial P_n, where $P_n = y_1$ for n even and $P_n = y_2$ for n odd. Only the case of n even, $n = 2N$, will be verified. The odd case is left as an exercise for the reader. Observe that $2r(2r+1) - n(n+1) = (2r-n)(n+2r+1)$, which implies the following relation for the coefficients of y_1:

$$\begin{aligned} c_{2p+2} &= c_0 \Pi_{r=0}^{p} \frac{2r(2r+1) - n(n+1)}{(2r+1)(2r+2)} \\ &= c_0 \Pi_{r=0}^{p} \frac{(2r-n)(n+2r+1)}{(2r+1)(2r+2)}. \end{aligned}$$

Choose

$$c_0 = \frac{(-1)^N}{2^n (N!)^2} \quad (n = 2N \text{ even}).$$

We will match coefficients in the reported formula for P_n against the series solution. The constant terms match by the choice of c_0. To match powers x^{n-2k} and x^{2p+2}, we require $n - 2k = 2p + 2$. To match coefficients, we must prove

$$c_{2p+2} = \frac{1}{2^n} \frac{(-1)^r (2n-2k)!}{k!(n-2k)!(n-k)!}.$$

Solving $n - 2k = 2p + 2$ for p when $n = 2N$ gives $p = N - k - 1$ and then

$$\begin{aligned} c_{2p+2} &= c_0 \Pi_{r=0}^{p} \frac{(-1)(n-2r)(n+2r+1)}{(2r+1)(2r+2)} \\ &= \frac{(-1)^{2N-k}}{2^n (N!)^2} \Pi_{r=0}^{N-k-1} \frac{(n-2r)(n+2r+1)}{(2r+1)(2r+2)}. \end{aligned}$$

The product factor will be converted to powers and factorials.

$$\begin{aligned} \boxed{\mathbf{1}} &= \Pi_{r=0}^{N-k-1} (n-2r) \\ &= (2N)(2N-2)\cdots(2k+2) \\ &= 2^{N-k}(N)(N-1)\cdots(k+1) \\ &= 2^{N-k} \frac{N!}{k!}. \end{aligned}$$

$$\begin{aligned}
\boxed{2} &= \Pi_{r=0}^{N-k-1}(n+2r+1) \\
&= (2N+1)(2N+3)\cdots(4N-2k-1) \\
&= \frac{(2N+1)(2N+2)(2N+3)(2N+4)\cdots(4N-2k-1)(4N-2k)}{(2N+2)(2N+4)\cdots(4N-2k)} \\
&= \frac{(4N-2k)!}{(2N)!(2N)(4N)\cdots(4N-2k)} \\
&= \frac{(4N-2k)!}{(2N)!2^{N-k}(N+1)(N+2)\cdots(2N-k)} \\
&= \frac{(4N-2k)!N!}{(2N)!2^{N-k}(2N-k)!} \\
&= \frac{(2n-2k)!N!}{(n)!2^{N-k}(n-k)!} \quad \text{because } n=2N. \\
\boxed{3} &= \Pi_{r=0}^{N-k-1}(2r+1)(2r+2) \\
&= [1\cdot 2][3\cdot 4]\cdots[(2N-2k-1)(2N-2k)] \\
&= (n-2k)! \quad \text{because } n=2N.
\end{aligned}$$

Then

$$\begin{aligned}
c_{2p+2} &= \frac{(-1)^{2N-k}}{2^n(N!)^2}\frac{\boxed{1}\,\boxed{2}}{\boxed{3}} \\
&= \frac{(-1)^{2N-k}}{2^n(N!)^2}\,\frac{2^{N-k}\dfrac{N!}{k!}\dfrac{(2n-2k)!N!}{(n)!2^{N-k}(n-k)!}}{(n-2k)!} \\
&= \frac{(-1)^k}{2^n k!(n-2k)!(n-k)!}.
\end{aligned}$$

This completes the derivation of the Legendre polynomial formula.

Derivation of Rodrigues' Formula. It must be shown that the expression for $P_n(x)$ is given by the expansion

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right).$$

Proof: Start with the binomial expansion $(a+b)^n = \sum_{k=0}^n \left(\begin{array}{c} n \\ k \end{array}\right) a^k b^{n-k}$. Substitute $a=-1$, $b=x^2$ to obtain

$$\begin{aligned}
(-1+x^2)^n &= \sum_{k=0}^n \left(\begin{array}{c} n \\ k \end{array}\right)(-1)^k(x^2)^{n-k} \\
&= \sum_{k=0}^n \frac{(-1)^k n!}{k!(n-k)!}x^{2n-2k}.
\end{aligned}$$

The plan is to differentiate this formula n times. The derivative $(d/du)^n u^m$ can be written as $\dfrac{m!}{(m-n)!}u^{m-n}$. Each differentiation annihilates the constant term. Therefore, there are $N = n/2$ terms for n even and $N = (n-1)/2$ terms for n odd, and we have

$$\begin{aligned} \frac{d^n}{dx^n}\left((-1+x^2)^n\right) &= \sum_{k=0}^{N} \frac{(-1)^k n!(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} \\ &= 2^n n! P_n(x). \end{aligned}$$

The proof is complete.

Exercises 12.7

Equivalent equations. Show that the given equation can be transformed to Legendre's differential equation.

1. $((1-x^2)y')' + n(n+1)y = 0$

2. Let $x = \cos\theta$, then $\sin\theta \dfrac{d^2y}{d\theta^2} + \cos\theta \dfrac{dy}{d\theta} + n(n+1)\sin\theta y = 0.$

Properties. Establish the given identity using the series relation for P_n or other identities.

3. $P_n(1) = 1$

4. $P_n(-1) = (-1)^n$

12.8 Orthogonality

The notion of orthogonality originates in $\mathcal{R}^3$, where nonzero vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are said to be **orthogonal**, written $\mathbf{v}_1 \perp \mathbf{v}_2$, provided $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. The dot product in $\mathcal{R}^3$ is defined by

$$\mathbf{x} \cdot \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Similarly, $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ defines the dot product in $\mathcal{R}^n$. Literature uses the notation $(\mathbf{x}, \mathbf{y})$ as well as $\mathbf{x} \cdot \mathbf{y}$. Modern terminology uses **inner product** instead of **dot product**, to emphasize the use of functions and abstract properties. The inner product satisfies the following properties.

$(\mathbf{x}, \mathbf{x}) \geq 0$	Non-negativity.
$(\mathbf{x}, \mathbf{x}) = 0$ implies $\mathbf{x} = \mathbf{0}$	Uniqueness.
$(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$	Symmetry.
$k(\mathbf{x}, \mathbf{y}) = (k\mathbf{x}, \mathbf{y})$	Homogeneity.
$(\mathbf{x} + \mathbf{y}, \mathbf{z}) = (\mathbf{x}, \mathbf{z}) + (\mathbf{y}, \mathbf{z})$	Additivity.

The storage system of choice for answers to differential equations is a real vector space V of functions f. A **real inner product space** is a vector space V with real-valued inner product function $(\mathbf{x}, \mathbf{y})$ defined for each $\mathbf{x}$, $\mathbf{y}$ in V, satisfying the preceding rules.

Dot Product for Functions. The extension of the notion of dot product to functions replaces $\mathbf{x} \cdot \mathbf{y}$ by average value. Insight can be gained from the approximation

$$\frac{1}{b-a} \int_a^b F(x) dx \approx \frac{F(x_1) + F(x_2) + \cdots + F(x_n)}{n}$$

where $b - a = nh$ and $x_k = a + kh$. The left side of this approximation is called the **average value of** F **on** $[a, b]$. The right side is the classical average of F at n equally spaced values in $[a, b]$. If we replace F by a product fg, then the average value formula reveals that $\int_a^b fg dx$ acts like a dot product:

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \approx \frac{\mathbf{x} \cdot \mathbf{y}}{n}, \quad \mathbf{x} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{pmatrix}.$$

The formula says that $\int_a^b f(x)g(x)dx$ is approximately a constant multiple of the dot product of samples of f, g at n points of $[a,b]$.

Given functions f and g integrable on $[a,b]$, the formula

$$(f,g) = \int_a^b f(x)g(x)dx$$

defines a dot product satisfying the abstract properties cited above. When dealing with solutions to differential equations, this dot product, along with the abstract properties of a dot product, provide the notions of *distance* and *orthogonality* analogous to those in $\mathcal{R}^3$.

Orthogonality, Norm and Distance. Define nonzero functions f and g to be **orthogonal** on $[a,b]$ provided $(f,g)=0$. Define the **norm** or the **distance** from f to 0 to be the number $\|f\| = \sqrt{(f,f)}$ and the distance from f to g to be $\|f-g\|$. The basic properties of the norm $\|\cdot\|$ are as follows.

$\|f\| \ge 0$	Non-negativity.
$\|f\| = 0$ implies $f = 0$	Uniqueness.
$\|f+g\| \le \|f\| + \|g\|$	The triangle inequality.
$\|cf\| = \|c\|\|f\|$	Homogeneity.
$\|f\| = \sqrt{(f,f)}$	Norm and the inner product.

Weighted Dot Product. In applications of Bessel functions, use is made of the **weighted dot product**

$$(f,g) = \int_a^b f(x)g(x)\rho(x)dx,$$

where $\rho(x) > 0$ on $a < x < b$.

The possibility that $\rho(x) = 0$ at some set of points in (a,b) has been considered by researchers, as well as the possibility of singularity at $x=a$ or $x=b$. Finally, $a=-\infty$ and/or $b=\infty$ have also been considered. Properties we advertise here mostly hold in these extended cases, provided appropriate additional assumptions are invoked.

Theorem 9 (Orthogonality of Legendre Polynomials)
The Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \quad n \ne m.$$

The relation means that P_n and P_m $(n \ne m)$ are orthogonal on $[-1,1]$ relative to the dot product $(f,g) = \int_{-1}^{1} f(x)g(x)dx$.

Proof: The details use only the Legendre differential equation $(1-x^2)y''-2xy'+n(n+1)y=0$ in the form $((1-x^2)y')'+n(n+1)y=0$ and the fact that $a(x)=1-x^2$ is zero at $x=\pm1$. From the definition of the Legendre polynomials, the following differential equations are satisfied:

$$\begin{aligned}(aP_n')'+n(n+1)P_n &= 0,\\ (aP_m')'+m(m+1)P_m &= 0.\end{aligned}$$

Multiply the first by P_m and the second by P_n, then subtract to obtain

$$(m(m+1)-n(n+1))P_nP_m=(aP_n')'P_m-(aP_m')'P_n.$$

Re-write the right side of this equation as $(aP_n'P_m-aP_m'P_n)'$. Then integrate over $-1<x<1$ to obtain

$$\begin{aligned}\text{LHS} &= (m(m+1)-n(n+1))\int_{-1}^{1}P_n(x)P_m(x)dx\\ &= (a(x)P_n'(x)P_m(x)-a(x)P_m'(x)P_n(x))|_{x=-1}^{x=1}\\ &= 0.\end{aligned}$$

The result is zero because $a(x)=1-x^2$ is zero at $x=1$ and $x=-1$. The dot product of P_n and P_m is zero, because $m(m+1)-n(n+1)\neq0$. The proof is complete.

Theorem 10 (Orthogonality of Bessel Functions)
Let the Bessel function J_n have positive zeros $\{b_{mn}\}_{m=1}^{\infty}$. Given $R>0$, define $f_m(r)=J_n(b_{mn}r/R)$. Then the following weighted orthogonality relation holds.

$$\int_0^R f_i(r)f_j(r)rdr=0,\quad i\neq j.$$

The relation means that f_i and f_j $(i\neq j)$ are orthogonal on $[0,R]$ relative to the weighted dot product $(f,g)=\int_0^R f(r)g(r)\rho(r)dr$, where $\rho(r)=r$.

Proof: The details depend entirely upon the Bessel differential equation of order n, $x^2y''+xy'+(x^2-n^2)y=0$, and the condition $y(b_{mn})=0$, valid when $y=J_n$. Let $\lambda=b_{mn}/R$ and change variables by $x=\lambda r$, $w(r)=y(\lambda r)$. Then w satisfies $dw/dr=y'(x)\lambda$, $d^2w/dr^2=y''(x)\lambda^2$ and the differential equation for y implies the equation

$$r^2\frac{d^2w}{dr^2}(r)+r\frac{dw}{dr}(r)+(\lambda^2r^2-n^2)w(r)=0.$$

Apply this change of variables to Bessel's equation or orders i and j. Then

$$\begin{aligned}r^2f_i''(r)+rf_i'(r)+(b_{in}^2r^2R^{-2}-n^2)f_i(r) &= 0,\\ r^2f_j''(r)+rf_j'(r)+(b_{jn}^2r^2R^{-2}-n^2)f_j(r) &= 0.\end{aligned}$$

Multiply the first equation by $f_j(r)$ and the second by $f_i(r)$, then subtract and divide by r to obtain

$$rf_i''f_j-rf_j''f_i+f_i'f_j-f_j'f_i+(b_{in}^2-b_{jn}^2)rR^{-2}f_if_j=0.$$

Because of the calculus identities $rw'' + w' = (rw')'$ and $(rw_1'w_2 - rw_2'w_2)' = (rw_1')'w_2 - (rw_2')'w_1$, this equation can be re-written in the form

$$(b_{jn}^2 - b_{in}^2)R^{-2}rf_if_j = (rf_i'f_j - rf_j'f_i)'.$$

Integrate this equation over $0 < r < R$. Then the right side evaluates to zero, because of the conditions $f_i(R) = f_j(R) = 0$. The left side evaluates to a nonzero multiple of $\int_0^R f_i(r)f_j(r)rdr$. Therefore, the weighted dot product of f_i and f_j is zero. This completes the proof.

Series of Orthogonal Functions. Let (f,g) denote a dot product defined for functions f, g. Especially, we include $(f,g) = \int_a^b fgdx$ and a weighted dot product $(f,g) = \int_a^b fg\rho dx$. Let $\{f_n\}$ be a sequence of nonzero functions orthogonal with respect to the dot product (f,g), that is,

$$(f_i, f_j) = 0, \quad i \neq j, \quad (f_i, f_i) > 0.$$

A **generalized Fourier series** is a convergent series of functions

$$F(x) = \sum_{n=1}^{\infty} c_n f_n(x).$$

The coefficients $\{c_n\}$ are called the **Fourier coefficients** of F. Convergence is taken in the sense of the norm $\|g\| = \sqrt{(g,g)}$:

$$F = \sum_{n=1}^{\infty} c_n f_n \quad \text{means} \quad \lim_{N\to\infty} \left\| \sum_{n=1}^{N} c_n f_n - F \right\| = 0.$$

For example, when $\|g\| = \sqrt{(g,g)}$ and $(f,g) = \int_a^b fgdx$, then series convergence is called **mean-square convergence** and it is defined by

$$\lim_{N\to\infty} \sqrt{\int_a^b \left| \sum_{n=1}^{N} c_n f_n(x) - F(x) \right|^2 dx} = 0.$$

Orthogonal series method. The coefficients $\{c_n\}$ in an orthogonal series are determined by a technique called the **orthogonal series method**, described in words as follows.

> The coefficient c_n in an orthogonal series is found by taking the dot product of the equation with the orthogonal function that multiplies c_n.

The details of the method:

$$(F, f_n) = \left(\sum_{k=1}^{\infty} c_k f_k, f_n \right) \qquad \text{Dot product the equation with } f_n.$$

$(F, f_n) = \sum_{k=1}^{\infty} c_k (f_k, f_n)$	Apply dot product properties.
$(F, f_n) = c_n (f_n, f_n)$	By orthogonality, just one term remains from the series on the right.

Division after the last step leads to the **Fourier coefficient formula**

$$c_n = \frac{(F, f_n)}{(f_n, f_n)}.$$

Bessel inequality and Parseval equality. Assume given a dot product (f, g) for an orthogonal series expansion

$$F(x) = \sum_{n=1}^{\infty} c_n f_n(x).$$

Bessel's inequality

$$\sum_{n=1}^{N} \frac{|(F, f_n)|^2}{\|f_n\|^2} \le \|F\|^2$$

is proved as follows. Let $N \ge 1$ be given and let $S_N = \sum_{n=1}^{N} c_n f_n$. Then

$(S_N, S_N) = \left(\sum_{n=1}^{N} c_n f_n, \sum_{k=1}^{N} c_k f_k \right)$	Definition of S_N.
$= \sum_{n=1}^{N} \sum_{k=1}^{N} c_n c_k (f_n, f_k)$	Linearity properties of the dot product.
$= \sum_{n=1}^{N} c_n c_n (f_n, f_n)$	Because $(f_n, f_k) = 0$ for $n \neq k$.
$= \sum_{n=1}^{N} \|c_n\|^2 \|f_n\|^2$	Because $\|g\|^2 = (g, g)$.
$(F, S_N) = \sum_{n=1}^{N} c_n (F, f_n)$	Linearity of the dot product.
$= \sum_{n=1}^{N} \|c_n\|^2 \|f_n\|^2$	Fourier coefficient formula.

Then

$0 \le \|F - S_N\|^2$	The norm is non-negative.
$= (F - S_N, F - S_N)$	Use $\|g\|^2 = (g, g)$.
$= (F, F) + (S_N, S_N) - 2(F, S_N)$	Dot product properties.

$= (F,F) - \sum_{n=1}^{N} |c_n|^2 \|f_n\|^2$ Apply previous formulas.

This proves

$$\sum_{n=1}^{N} |c_n|^2 \|f_n\|^2 \le (F,F),$$

or what is the same, because of the Fourier coefficient formula,

$$\sum_{n=1}^{N} \frac{|(F,f_n)|^2}{\|f_n\|^2} \le (F,F).$$

Letting $N \to \infty$ gives Bessel's inequality $\sum_{n=1}^{\infty} \frac{|(F,f_n)|^2}{\|f_n\|^2} \le (F,F)$.

Parseval's equality is equality in Bessel's inequality:

$$\|F\|^2 = \sum_{n=1}^{N} \frac{|(F,f_n)|^2}{\|f_n\|^2}.$$

There is a fundamental relationship between Parseval's equality and the possibility to expand a function F as an infinite orthogonal series in the functions $\{f_n\}$. In literature, the relationship is known as **completeness** of the orthogonal sequence $\{f_n\}$. The definition: $\{f_n\}$ is complete if and only if each function F has a series expansion $F = \sum_{n=1}^{\infty} c_n f_n$ for some set of coefficients $\{c_n\}$. When equality holds, the coefficients c_n are given by Fourier's coefficient formula.

Theorem 11 (Parseval)
A sequence $\{f_n\}$ is a complete orthogonal sequence if and only if Parseval's equality holds.

Therefore, the equation $F = \sum_{n=1}^{\infty} \frac{(F,f_n)}{(f_n,f_n)} f_n$ holds for every F if and only if Parseval's equality holds for every F.

Legendre series. A convergent series of the form

$$F(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

is called a **Legendre series**. The orthogonal system $\{P_n\}$ on $[-1,1]$ under the dot product $(f,g) = \int_{-1}^{1} f(x)g(x)dx$ together with Fourier's coefficient formula gives

$$c_n = \frac{\int_{-1}^{1} F(x)P_n(x)dx}{\int_{-1}^{1} |P_n(x)|^2 dx}.$$

The denominator in this fraction can be evaluated for all values of n:

$$\int_{-1}^{1} |P_n(x)|^2 dx = \frac{2}{2n+1}.$$

Theorem 12 (Legendre expansion)
Let F be defined on $-1 \le x \le 1$ and assume F and F' are piecewise continuous. Then the Legendre series expansion of F converges and equals $F(x)$ at each point of continuity of F. At other points, the series converges to $\frac{1}{2}(F(x+) + F(x-))$.

Bessel series. A convergent infinite series of the form

$$F(r) = \sum_{n=1}^{\infty} c_n J_m(b_{nm} r/R), \quad 0 < r < R,$$

is called a **Bessel series**. The index m, assumed here to be a non-negative integer, is fixed throughout the series terms. The sequence $\{b_{nm}\}_{n=1}^{\infty}$ is an ordered list of the positive zeros of J_m.

The weighted dot product $(f,g) = \int_0^R f(r)g(r)rdr$ is used. It is known that the sequence of functions $f_n(r) = J_m(b_{nm}r/R)$ is orthogonal relative to the weighted dot product $(\cdot,\cdot)$. Then Fourier's coefficient formula implies

$$c_n = \frac{\int_0^R F(r)J_m(b_{nm}r/R)rdr}{\int_0^R |J_m(b_{nm}r/R)|^2 rdr}.$$

To evaluate the denominator of the above fraction, let's denote $' = d/dr$, $y(r) = f_n(r) = J_m(b_{nm}r/R)$. Use $r(ry')' + (b_{nm}^2 r^2 R^{-2} - n^2)y = 0$, the equation used to prove orthogonality of Bessel functions. Multiply this equation by $2y'$. Re-write the resulting equation as

$$[(ry')^2]' + (b_{nm}^2 r^2 R^{-2} - n^2)[y^2]' = 0.$$

Integrate this last equation over $[0,R]$. Use parts on the term involving $r^2[y^2]'$. Then use $J_m(0) = 0$, $y' = (b_{nm}/R)J_m'(b_{nm}r/R)$ and $xJ_m'(x) = mJ_m(x) - xJ_{m+1}(x)$ to obtain

$$\int_0^R |J_m(b_{nm}r/R)|^2 rdr = \frac{R^2}{2}|J_{m+1}(b_{nm})|^2.$$

Theorem 13 (Bessel expansion)
Let F be defined on $0 \le x \le R$ and assume F and F' are piecewise continuous. Then the Bessel series expansion of F converges and equals $F(x)$ at each point of continuity of F. At other points, the series converges to $\frac{1}{2}(F(x+) + F(x-))$.

Exercises 12.8

Legendre series. Establish the following results, using the supplied hints.

1. Prove $\int_{-1}^{1} |P_n(x)|^2 dx = \frac{2}{2n+1}$.

2.

3.

4.

5. Let $(f, g) = \int_0^\pi f(x)g(x)\sin(x)dx$. Show that the sequence

$$\{P_n(\cos x)\}$$

is orthogonal on $0 \le x \le \pi$ with respect to inner product (f, g).

6. Let $F(x) = \sin^3(x) - \sin(x)\cos(x)$. Expand F as a Legendre series of the form

$$F(x) = \sum_{n=0}^{\infty} c_n P_n(\cos x).$$

Chebyshev Polynomials. Define

$$T_n(x) = \cos(n \arccos(x)).$$

These are called **Chebyshev polynomials**. Define

$$(f, g) = \int_{-1}^{1} f(x)g(x)(1 - x^2)^{-1/2}dx.$$

7. Show that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$.

8. Show that $T_3(x) = 4x^3 - 3x$.

9. Prove that (f, g) satisfies the abstract properties of an inner product.

10. Show that T_n is a solution of the **Chebyshev equation** $(1 - x^2)y'' - xy' + n^2y = 0$.

11. Prove that $\{T_n\}$ is orthogonal relative to the weighted inner product (f, g).

Hermite Polynomials. Define the Hermite polynomials by $H_0(x) = 1$ and inductively

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right).$$

Define the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2/2}dx.$$

12. Show that $H_1(x) = x$, $H_2(x) = x^2 - 1$.

13. Show that $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$.

14. Prove that $H_n'(x) = nH_{n-1}(x)$.

15. Prove that $H_{n+1}(x) = xH_n(x) - H_n'(x)$.

16. Show that H_n is a polynomial solution of **Hermite's equation** $y'' - xy' + ny = 0$.

17. Prove that (f, g) defines an inner product.

18. Show that the sequence $\{H_n(x)\}$ is orthogonal with respect to (f, g).

Laguerre Polynomials. Define the Laguerre polynomials by$L_0(x) = 1$ and inductively

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

Define the inner product

$$(f, g) = \int_0^{\infty} f(x)g(x)e^{-x}dx.$$

19. Show that $L_1(x) = 1 - x$, $L_2(x) = 1 - 2x + x^2/2$.

20. Show that $L_3(x) = 1 - 3x + 3x^2/2 - x^3/6$.

21. Prove that (f, g) satisfies the abstract properties for an inner product.

22. Show that L_0, L_1, L_2 are orthogonal with respect to the inner product (f, g), using direct integration methods.

23. Show that L_n can be expressed by the formula

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!k!}(-x)^n.$$

24. Show that $\{L_n\}$ is an orthogonal sequence with respect to (f, g).

25. Find an expression for a polynomial solution to **Laguerre's equation** $xy'' + (1-x)y' + ny = 0$ using Frobenius theory.

26. Show that **Laguerre's equation** is satisfied for $y = L_n$.