Chapter 8

Fourier Transforms

Fourier series and their ilk are designed to solve boundary value problems on bounded intervals. The extension of the Fourier calculus to the entire real line leads naturally to the *Fourier transform*, a powerful mathematical tool for the analysis of non-periodic functions. The Fourier transform is of fundamental importance in a remarkably broad range of applications, including both ordinary and partial differential equations, probability, quantum mechanics, signal and image processing, and control theory, to name but a few.

In this chapter, we motivate the construction by investigating how (rescaled) Fourier series behave as the length of the interval goes to infinity. The resulting Fourier transform maps a function defined on physical space to a function defined on the space of frequencies, whose values quantify the "amount" of each periodic frequency contained in the original function. The inverse Fourier transform then reconstructs the original function from its transformed frequency components. The integrals defining the Fourier transform and its inverse are, remarkably, almost identical, and this symmetry is often exploited, for example when assembling tables of Fourier transforms.

One of the most important properties of the Fourier transform is that it converts calculus: differentiation and integration — into algebra: multiplication and division. This underlies its application to linear ordinary differential equations and, in the following chapters, partial differential equations. In engineering applications, the Fourier transform is sometimes overshadowed by the Laplace transform, which is a particular subcase. The Fourier transform is used to analyze boundary value problems on the entire line. The Laplace transform is better suited to solving initial value problems, [**24**], but will not be developed in this text.

The Fourier transform is, like Fourier series, completely compatible with the calculus of generalized functions, [**74**]. The final section contains a brief introduction to the analytical foundations of the subject, including the basics of Hilbert space. However, a full, rigorous development requires more powerful analytical tools, including the Lebesgue integral and complex analysis, and the interested reader is therefore referred to more advanced texts, including [**39**, **74**, **103**, **122**].

8.1. The Fourier Transform.

We begin by motivating the Fourier transform as a limiting case of Fourier series. Although the rigorous details are subtle, the underlying idea can be straightforwardly explained. Let $f(x)$ be a function defined for all $-\infty < x < \infty$. The goal is to construct a Fourier expansion for $f(x)$ in terms of basic trigonometric functions. One evident approach is to construct its Fourier series on progressively longer and longer intervals, and then take the limit as their lengths go to infinity. This limiting process converts the Fourier sums into integrals, and the resulting representation of a function is renamed the Fourier transform. Since we are dealing with an infinite interval, there are no longer any periodicity requirements on the function $f(x)$. Moreover, the frequencies represented in the Fourier transform are no longer constrained by the length of the interval, and so we are effectively decomposing a quite general, non-periodic function into a continuous superposition of trigonometric functions of all possible frequencies.

Let us present the details in a more concrete form. The computations will be significantly simpler if we work with the complex version of the Fourier series from the outset. Our starting point is the rescaled Fourier series (3.86) on a symmetric interval $[-\ell, \ell]$ of length 2ℓ, which we rewrite in the adapted form

$$f(x) \sim \sum_{\nu=-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \, \frac{\widehat{f}_\ell(k_\nu)}{\ell} \, e^{\,\mathrm{i}\, k_\nu x}. \tag{8.1}$$

The sum is over the discrete collection of frequencies

$$k_\nu = \frac{\pi\nu}{\ell}, \qquad\qquad \nu = 0, \pm 1, \pm 2, \ldots\,, \tag{8.2}$$

corresponding to those trigonometric functions that have period 2ℓ. For reasons that will soon become apparent, the Fourier coefficients of f are now denoted as

$$c_\nu = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)\, e^{-\,\mathrm{i}\, k_\nu x}\, dx = \sqrt{\frac{\pi}{2}} \, \frac{\widehat{f}_\ell(k_\nu)}{\ell}, \tag{8.3}$$

so that

$$\widehat{f}_\ell(k_\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\ell}^{\ell} f(x)\, e^{-\,\mathrm{i}\, k_\nu x}\, dx. \tag{8.4}$$

This reformulation of the basic Fourier series formula allows us to easily pass to the limit when the interval's length $\ell \to \infty$.

On an interval of length 2ℓ, the frequencies (8.2) required to represent a function in Fourier series form are equally distributed, with interfrequency spacing

$$\Delta k = k_{\nu+1} - k_\nu = \frac{\pi}{\ell}\,. \tag{8.5}$$

As $\ell \to \infty$, the spacing $\Delta k \to 0$, and so the relevant frequencies become more and more densely packed in the line $-\infty < k < \infty$. In the limit, we thus anticipate that *all* possible frequencies will be represented. Indeed, letting $k_\nu = k$ be arbitrary in (8.4), and sending $\ell \to \infty$, results in the infinite integral

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-\,\mathrm{i}\, k x}\, dx \tag{8.6}$$

known as the *Fourier transform* of the function $f(x)$. If $f(x)$ is a sufficiently nice function, e.g., piecewise continuous and decaying to 0 reasonably quickly as $|\,x\,| \to \infty$, its Fourier transform $\widehat{f}(k)$ is defined for all possible frequencies $k \in \mathbb{R}$. The preceding formula will sometimes conveniently be abbreviated as

$$\widehat{f}(k) = \mathcal{F}[\,f(x)\,], \tag{8.7}$$

where $\mathcal{F}$ is the *Fourier transform operator*, that maps each (sufficiently nice) function of the spatial variable x to a function of the frequency variable k.

To reconstruct the function from its Fourier transform, we apply a similar limiting procedure to the Fourier series (8.1), which we first rewrite in a more suggestive form,

$$f(x) \;\sim\; \frac{1}{\sqrt{2\pi}} \sum_{\nu=-\infty}^{\infty} \widehat{f}_\ell(k_\nu)\, e^{\,\mathrm{i}\, k_\nu x}\, \Delta k, \tag{8.8}$$

using (8.5). For each fixed value of x, the right hand side has the form of a Riemann sum approximating the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}_\ell(k)\, e^{\,\mathrm{i}\, k x}\, dk.$$

As $\ell \to \infty$, the functions (8.4) converge to the Fourier transform: $\widehat{f}_\ell(k) \to \widehat{f}(k)$; moreover, the interfrequency spacing $\Delta k = \pi/\ell \to 0$, and so one expects the Riemann sums to converge to the limiting integral

$$f(x) \;\sim\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\, k x}\, dk. \tag{8.9}$$

The resulting formula serves to define the *inverse Fourier transform*, which is used to recover the original signal from its Fourier transform. In this manner, the Fourier series has become a Fourier integral that reconstructs the function $f(x)$ as a (continuous) superposition of complex exponentials $e^{\,\mathrm{i}\, k x}$ of *all* possible frequencies, with $\widehat{f}(k)/\sqrt{2\pi}$ quantifying the amount contributed by the complex exponential of frequency k. In abbreviated form, formula (8.9) can be written

$$f(x) = \mathcal{F}^{-1}[\,\widehat{f}(k)\,], \tag{8.10}$$

thus defining the inverse of the Fourier transform operator (8.7).

It is worth pointing out that both the Fourier transform (8.7) and its inverse (8.10) define linear maps on function space. This means that the Fourier transform of the sum of two functions is the sum of their individual transforms, while multiplying a function by a constant multiplies its Fourier transform by the same factor:

$$\begin{aligned} \mathcal{F}[\,f(x) + g(x)\,] &= \mathcal{F}[\,f(x)\,] + \mathcal{F}[\,g(x)\,] = \widehat{f}(k) + \widehat{g}(k), \\ \mathcal{F}[\,c\,f(x)\,] &= c\,\mathcal{F}[\,f(x)\,] = c\,\widehat{f}(k). \end{aligned} \tag{8.11}$$

A similar statement holds for the inverse Fourier transform $\mathcal{F}^{-1}$.

Recapitulating, by letting the length of the interval go to ∞, the discrete Fourier series has become a continuous Fourier integral, while the Fourier coefficients, which were defined only at a discrete collection of possible frequencies, have become a complete function $\widehat{f}(k)$ defined on all of frequency space $k \in \mathbb{R}$. The reconstruction of $f(x)$ from its Fourier transform $\widehat{f}(k)$ via (8.9) can be rigorously justified under suitable hypotheses. For example, if $f(x)$ is piecewise C^1 on all of $\mathbb{R}$ and decays reasonably rapidly, $f(x) \to 0$ as $|\,x\,| \to \infty$, in order that its Fourier integral (8.6) converges absolutely, then it can be proved, [**39, 122**], that the inverse Fourier integral (8.9) will converge to $f(x)$ at all points of continuity, and to the midpoint $\frac{1}{2}(f(x^-) + f(x^+))$ at jump discontinuities — just like a Fourier series. In particular, its Fourier transform $\widehat{f}(k) \to 0$ must also decay as $|\,k\,| \to \infty$, implying that (as with Fourier series) the very high frequency modes make negligible contributions to the reconstruction of such a signal. A more precise result will be formulated in Theorem 8.15 below.

Example 8.1. The Fourier transform of the rectangular pulse[†]

$$f(x) = \sigma(x+a) - \sigma(x-a) = \begin{cases} 1, & -a < x < a, \\ 0, & |\,x\,| > a, \end{cases} \tag{8.12}$$

or *box function*, of width $2a$, is easily computed:

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-\,\mathrm{i}\,k\,x}\,dx = \frac{e^{\,\mathrm{i}\,k\,a} - e^{-\,\mathrm{i}\,k\,a}}{\sqrt{2\pi}\;\mathrm{i}\,k} = \sqrt{\frac{2}{\pi}}\,\frac{\sin a\,k}{k}\,. \tag{8.13}$$

On the other hand, the reconstruction of the pulse via the inverse transform (8.9) tells us that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{\,\mathrm{i}\,k\,x} \sin a\,k}{k}\,dk = f(x) = \begin{cases} 1, & -a < x < a, \\ \frac{1}{2}, & x = \pm\,a, \\ 0, & |\,x\,| > a. \end{cases} \tag{8.14}$$

Note the convergence to the middle of the jump discontinuities at $x = \pm\,a$. The real part of this complex integral produces a striking trigonometric integral identity

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos x\,k\;\sin a\,k}{k}\,dk = \begin{cases} 1, & -a < x < a, \\ \frac{1}{2}, & x = \pm\,a, \\ 0, & |\,x\,| > a. \end{cases} \tag{8.15}$$

Just as many Fourier series yield nontrivial summation formulae, the reconstruction of a function from its Fourier transform often leads to nontrivial integration formulas. One *cannot* compute the integral (8.14) by the Fundamental Theorem of Calculus, since there is no elementary function whose derivative equals the integrand[‡]. In Figure 8.1 we display the box function with $a = 1$, its Fourier transform, along with a reconstruction obtained by numerically integrating (8.15). Since we are dealing with an infinite integral, we must break off the numerical integrator by restricting it to a finite interval. The first graph is obtained by integrating from $-5 \le k \le 5$ while the second is from $-10 \le k \le 10$. The

[†] $\sigma(x)$ is the step function (3.46).

[‡] One can use Euler's formula (3.59) to reduce (8.14) to a complex version of the *exponential integral* $\int (e^{\alpha\,k}/k)\,dk$, but it can be proved, [**26**], that neither integral can be written in terms of elementary functions.

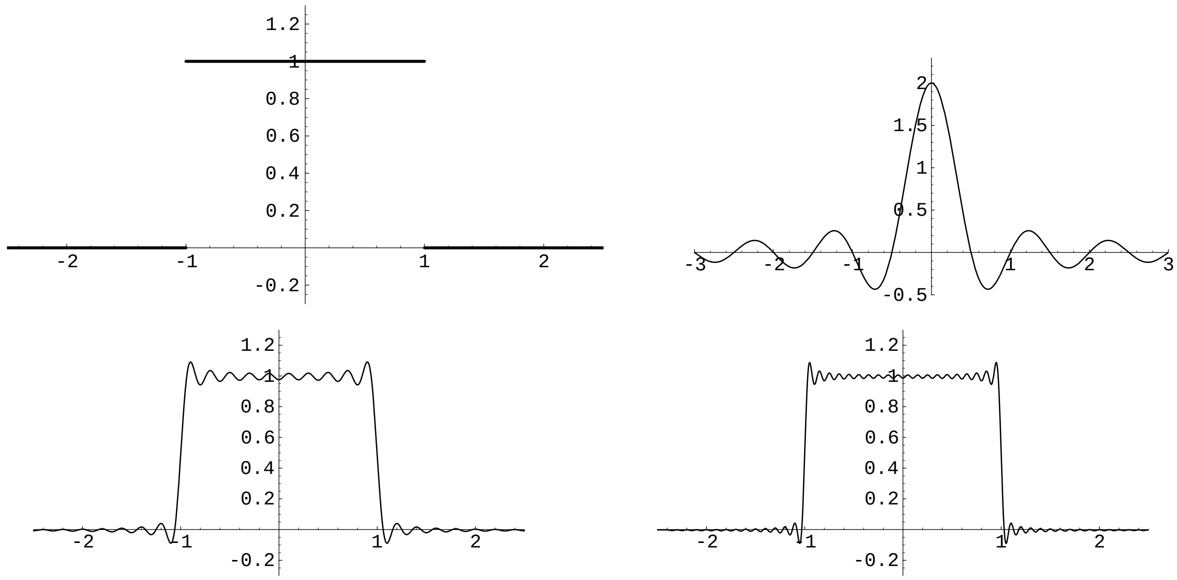


Figure 8.1. Fourier Transform of a Rectangular Pulse.

non-uniform convergence of the integral leads to the appearance of a Gibbs phenomenon at the two discontinuities, similar to what we observed in the non-uniform convergence of a Fourier series.

On the other hand, the identity resulting from the imaginary part,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin k\,x\ \sin a\,k}{k}\,dk=0,$$

is, on the surface, not surprising because the integrand is odd. However, it is far from obvious that either integral converges; indeed, the amplitude of the oscillatory integrand decays like $1/|\,k\,|$, but the latter function does not have a convergent integral, and so the usual comparison test for infinite integrals, [**8**, **102**], fails to apply. Their convergence is marginal at best, and the trigonometric oscillations somehow manage to ameliorate the slow rate of decay of $1/k$.

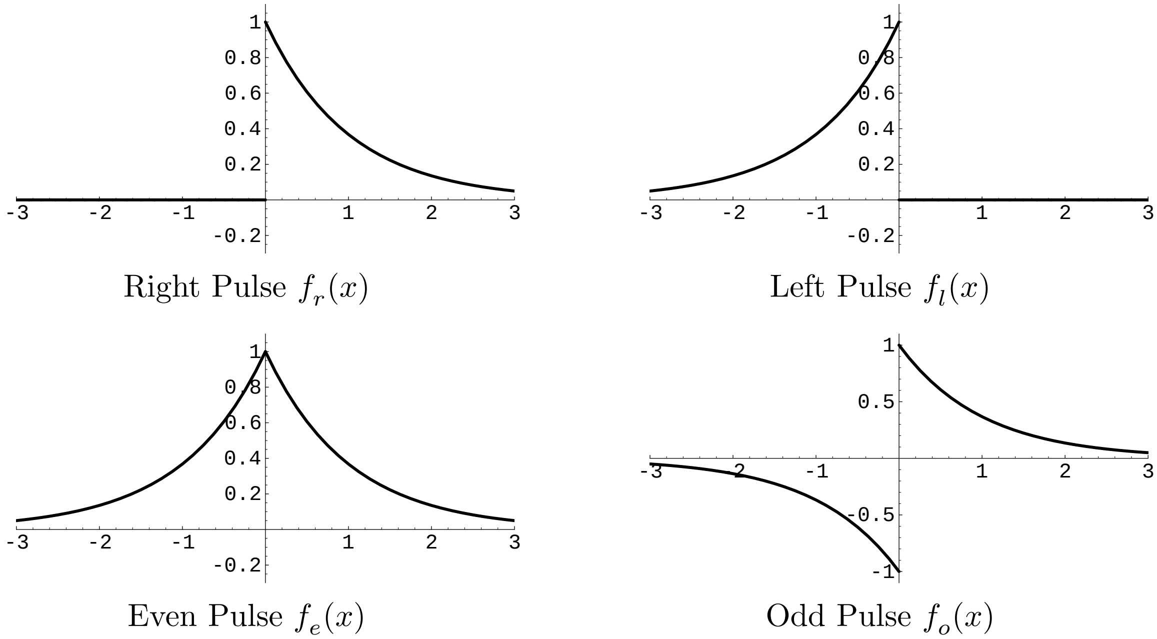
Example 8.2. Consider an exponentially decaying right-handed pulse[†]

$$f_r(x)=\begin{cases} e^{-a\,x}, & x>0,\\ 0, & x<0,\end{cases} \tag{8.16}$$

where $a>0$. We compute its Fourier transform directly from the definition:

$$\widehat{f}_r(k)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-a\,x}\,e^{-\,\mathrm{i}\,k\,x}\,dx=-\,\frac{1}{\sqrt{2\pi}}\,\frac{e^{-(a+\mathrm{i}\,k)x}}{a+\mathrm{i}\,k}\,\Bigg|_{x=0}^{\infty}=\frac{1}{\sqrt{2\pi}\,(a+\mathrm{i}\,k)}\,.$$

[†] Note that we can't Fourier transform the entire exponential function $e^{-\,a\,x}$ because it does not go to zero at both $\pm\infty$, which is required for the integral (8.6) to converge.



Right Pulse $f_r(x)$ Left Pulse $f_l(x)$

Even Pulse $f_e(x)$ Odd Pulse $f_o(x)$

Figure 8.2. Exponential Pulses.

As in the preceding example, the inverse Fourier transform produces a nontrivial integral identity:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\,\mathrm{i}\,k\,x}}{a+\mathrm{i}\,k}\,dk=\begin{cases} e^{-a\,x}, & x>0,\\ \tfrac{1}{2}\,, & x=0,\\ 0, & x<0.\end{cases} \tag{8.17}$$

Similarly, a pulse that decays to the left,

$$f_l(x)=\begin{cases} e^{a\,x}, & x<0,\\ 0, & x>0,\end{cases} \tag{8.18}$$

where $a>0$ is still positive, has Fourier transform

$$\widehat{f}_l(k)=\frac{1}{\sqrt{2\pi}\,(a-\mathrm{i}\,k)}\,. \tag{8.19}$$

This also follows from the general fact that the Fourier transform of $f(-x)$ is $\widehat{f}(-k)$; see Exercise ■. The even exponentially decaying pulse

$$f_e(x)=e^{-a\,|\,x\,|} \tag{8.20}$$

is merely the sum of left and right pulses: $f_e=f_r+f_l$. Thus, by linearity,

$$\widehat{f}_e(k)=\widehat{f}_r(k)+\widehat{f}_l(k)=\frac{1}{\sqrt{2\pi}\,(a+\mathrm{i}\,k)}\;+\;\frac{1}{\sqrt{2\pi}\,(a-\mathrm{i}\,k)}=\sqrt{\frac{2}{\pi}}\,\frac{a}{k^2+a^2}\,, \tag{8.21}$$

The resulting Fourier transform is real and even because $f_e(x)$ is a real-valued even function; see Exercise ■. The inverse Fourier transform (8.9) produces another nontrivial integral identity:

$$e^{-a\,|\,x\,|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,e^{\,\mathrm{i}\,k\,x}}{k^2+a^2}\,dk = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos k\,x}{k^2+a^2}\;dk. \tag{8.22}$$

(The imaginary part of the integral vanishes because its integrand is odd.) On the other hand, the odd exponentially decaying pulse,

$$f_o(x) = (\operatorname{sign} x)\,e^{-a\,|\,x\,|} = \begin{cases} e^{-a\,x}, & x > 0, \\ -\,e^{a\,x}, & x < 0, \end{cases} \tag{8.23}$$

is the difference of the right and left pulses, $f_o = f_r - f_l$, and has purely imaginary and odd Fourier transform

$$\widehat{f}_o(k) = \widehat{f}_r(k) - \widehat{f}_l(k) = \frac{1}{\sqrt{2\pi}\,(a+\mathrm{i}\,k)} - \frac{1}{\sqrt{2\pi}\,(a-\mathrm{i}\,k)} = -\,\mathrm{i}\,\sqrt{\frac{2}{\pi}}\,\frac{k}{k^2+a^2}\,. \tag{8.24}$$

The inverse transform is

$$(\operatorname{sign} x)\,e^{-a\,|\,x\,|} = -\,\frac{\mathrm{i}}{\pi} \int_{-\infty}^{\infty} \frac{k\,e^{\,\mathrm{i}\,k\,x}}{k^2+a^2}\,dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{k\sin k\,x}{k^2+a^2}\;dk. \tag{8.25}$$

As a final example, consider the rational function

$$f(x) = \frac{1}{x^2+a^2}\,, \qquad \text{where} \qquad a > 0. \tag{8.26}$$

Its Fourier transform requires integrating

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\,\mathrm{i}\,k\,x}}{x^2+a^2}\;dx. \tag{8.27}$$

The indefinite integral (anti-derivative) does not appear in basic integration tables, and, in fact, cannot be done in terms of elementary functions. However, we have just managed to evaluate this particular integral! Look at (8.22). If we change x to k and k to $-x$, then we exactly recover the integral (8.27) up to a factor of $a\sqrt{2/\pi}$. We conclude that the Fourier transform of (8.26) is

$$\widehat{f}(k) = \sqrt{\frac{\pi}{2}}\;\frac{e^{-a\,|\,k\,|}}{a}\,. \tag{8.28}$$

This last example is indicative of an important general fact. The reader has no doubt already noted the remarkable similarity between the Fourier transform (8.6) and its inverse (8.9). Indeed, the only difference is that the former has a minus sign in the exponential. This implies the following *Symmetry Principle* relating the direct and inverse Fourier transforms.

Theorem 8.3. *If the Fourier transform of the function $f(x)$ is $\widehat{f}(k)$, then the Fourier transform of $\widehat{f}(x)$ is $f(-k)$.*

The Symmetry Principle allows us to reduce the tabulation of Fourier transforms by half. For instance, referring back to Example 8.1, we deduce that the Fourier transform of the function

$$f(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin a\,x}{x}$$

is

$$\widehat{f}(k) = \sigma(-k+a) - \sigma(-k-a) = \sigma(k+a) - \sigma(k-a) = \begin{cases} 1, & -a < k < a, \\ \frac{1}{2}, & k = \pm a, \\ 0, & |\,k\,| > a. \end{cases} \tag{8.29}$$

Note that, by linearity, we can divide both $f(x)$ and $\widehat{f}(k)$ by $\sqrt{2/\pi}$ to deduce the Fourier transform of $\dfrac{\sin a\,x}{x}$.

Warning: Some authors omit the $\sqrt{2\pi}$ factor in the definition (8.6) of the Fourier transform $\widehat{f}(k)$. This alternative convention does have a slight advantage of eliminating many $\sqrt{2\pi}$ factors in the Fourier transform expressions. However, this necessitates an extra such factor in the reconstruction formula (8.9), which is achieved by replacing $\sqrt{2\pi}$ by 2π. A significant disadvantage is that the resulting formulae for the Fourier transform and its inverse are less similar, and so the Symmetry Principle of Theorem 8.3 requires some modification. (On the other hand, convolution — to be discussed below — is a little easier without the extra factor.) Yet another, more recent convention can be found in Exercise ■. When consulting any particular reference, the reader *always* needs to check which version of the Fourier transform is being used.

All of the functions in Example 8.2 required $a > 0$ for the Fourier integrals to converge. The functions that emerge in the limit as a goes to 0 are of special interest. Let us start with the odd exponential pulse (8.23). When $a \to 0$, the function $f_o(x)$ converges to the *sign function*

$$f(x) = \operatorname{sign} x = \sigma(x) - \sigma(-x) = \begin{cases} +1, & x > 0, \\ -1, & x < 0. \end{cases} \tag{8.30}$$

Taking the limit of the Fourier transform (8.24) leads to

$$\widehat{f}(k) = -\,\mathrm{i}\,\sqrt{\frac{2}{\pi}}\,\frac{1}{k}\,. \tag{8.31}$$

The nonintegrable singularity of $\widehat{f}(k)$ at $k = 0$ is indicative of the fact that the sign function does *not* decay as $|\,x\,| \to \infty$. In this case, neither the Fourier transform integral nor its inverse are well-defined as standard (Riemann, or even Lebesgue) integrals. Nevertheless, it is possible to rigorously justify these results within the framework of generalized functions.

More interesting are the even pulse functions $f_e(x)$, which, in the limit $a \to 0$, become the constant function

$$f(x) \equiv 1. \tag{8.32}$$

The limit of the Fourier transform (8.21) is

$$\lim_{a \to 0} \sqrt{\frac{2}{\pi}}\,\frac{2\,a}{k^2 + a^2} = \begin{cases} 0, & k \neq 0, \\ \infty, & k = 0. \end{cases} \tag{8.33}$$

This limiting behavior should remind the reader of our construction (6.10) of the delta function as the limit of the functions

$$\delta(x) = \lim_{n\to\infty} \frac{n}{\pi\,(1+n^2x^2)} = \lim_{a\to 0} \frac{a}{\pi\,(a^2+x^2)}\,.$$

Comparing with (8.33), we conclude that the Fourier transform of the constant function (8.32) is a multiple of the delta function in the frequency variable:

$$\widehat{f}(k) = \sqrt{2\pi}\;\delta(k). \tag{8.34}$$

The direct transform integral

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\,\mathrm{i}\,k\,x}\,dx \tag{8.35}$$

is, strictly speaking, not defined because the infinite integrals of the oscillatory sine and cosine functions don't converge! However, this identity can be validly interpreted within the framework of weak convergence and generalized functions. On the other hand, the inverse transform formula (8.9) yields

$$\int_{-\infty}^{\infty} \delta(k)\,e^{\,\mathrm{i}\,k\,x}\,dk = e^{\,\mathrm{i}\,k\,0} = 1,$$

which is in accord with the basic definition (6.16) of the delta function. As in the preceding case, the delta function singularity at $k=0$ manifests the lack of decay of the constant function.

Conversely, the delta function $\delta(x)$ has constant Fourier transform

$$\widehat{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x)\,e^{-\,\mathrm{i}\,k\,x}\,dx = \frac{e^{-\,\mathrm{i}\,k\,0}}{\sqrt{2\pi}} \equiv \frac{1}{\sqrt{2\pi}}\,, \tag{8.36}$$

a result that also follows from the Symmetry Principle of Theorem 8.3. To determine the Fourier transform of a delta spike $\delta_\xi(x) = \delta(x-\xi)$ concentrated at position $x=\xi$, we compute

$$\widehat{\delta_\xi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-\xi)\,e^{-\,\mathrm{i}\,k\,x}\,dx = \frac{e^{-\,\mathrm{i}\,k\,\xi}}{\sqrt{2\pi}}\,. \tag{8.37}$$

The result is a pure exponential in frequency space. Applying the inverse Fourier transform (8.9) leads, at least on a formal level, to the remarkable identity

$$\delta_\xi(x) = \delta(x-\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,k(x-\xi)}\,dk = \frac{1}{2\pi}\,\langle\, e^{\,\mathrm{i}\,k\,x}\,; e^{\,\mathrm{i}\,k\,\xi}\,\rangle\,, \tag{8.38}$$

where $\langle\,\cdot\,;\cdot\,\rangle$ denotes the L^2 Hermitian inner product of complex-valued functions of $k \in \mathbb{R}$. Since the delta function vanishes for $x \neq \xi$, this identity is telling us that complex exponentials of differing frequencies are mutually orthogonal. However, as with (8.35), this only make sense within the language of generalized functions. On the other hand, multiplying both sides of (8.38) by $f(\xi)$, and then integrating with respect to ξ produces

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi)\,e^{\,\mathrm{i}\,k(x-\xi)}\,dx\,dk. \tag{8.39}$$

This *is* a perfectly valid formula, being a restatement (or, rather, combination) of the basic formulae (8.6) and (8.9) connecting the direct and inverse Fourier transforms of the function $f(x)$.

Vice versa, the Symmetry Principle tells us that the Fourier transform of a pure exponential $e^{\,\mathrm{i}\,\kappa x}$ will be a shifted delta spike $\sqrt{2\pi}\,\delta(k-\kappa)$, concentrated at frequency $k=\kappa$. Both results are particular cases of the following Shift Theorem, whose proof is left as an exercise for the reader.

Theorem 8.4. *If $f(x)$ has Fourier transform $\widehat{f}(k)$, then the Fourier transform of the shifted function $f(x-\xi)$ is $e^{-\,\mathrm{i}\,k\xi}\,\widehat{f}(k)$. Similarly, the transform of the product function $e^{\,\mathrm{i}\,\kappa x}\,f(x)$, for real κ, is the shifted transform $\widehat{f}(k-\kappa)$.*

In a similar vein, the Dilation Theorem gives the effect of a scaling transformation on the Fourier transform. Again, the proof is left to the reader.

Theorem 8.5. *If $f(x)$ has Fourier transform $\widehat{f}(k)$, then the Fourier transform of the rescaled function $f(cx)$ for $0 \neq c \in \mathbb{R}$ is $\dfrac{1}{|\,c\,|}\,\widehat{f}\left(\dfrac{k}{c}\right)$.*

Example 8.6. Let us determine the Fourier transform of the Gaussian function $g(x)=e^{-x^2}$. To evaluate its Fourier integral, we first complete the square in the exponent:

$$\begin{aligned}\widehat{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2-\,\mathrm{i}\,kx}\,dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-\,\mathrm{i}\,k/2)^2-k^2/4}\,dx \\ &= \frac{e^{-k^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{e^{-k^2/4}}{\sqrt{2}}\,.\end{aligned}$$

The next to last equality employed the change of variables† $y = x - \frac{1}{2}\,\mathrm{i}\,k$, while the final step used formula (Gaussint■).

More generally, to find the Fourier transform of $g_a(x) = e^{-a x^2}$, where $a>0$, we invoke the Dilation Theorem 8.5 with $c=\sqrt{a}$ to deduce that $\widehat{g}_a(k) = e^{-k^2/(4a)}/\sqrt{2a}$.

Since the Fourier transform uniquely associates a function $\widehat{f}(k)$ on frequency space with each (reasonable) function $f(x)$ on physical space, one can characterize functions by their transforms. Many practical applications rely on tables (or, even better, computer algebra systems such as MATHEMATICA or MAPLE) that recognize a wide variety of transforms of basic functions of importance in applications. The accompanying table lists some of the most important examples of functions and their Fourier transforms, based on our convention (8.6). Keep in mind that, by applying the Symmetry Principle of Theorem 8.3, each entry can be used to deduce two different Fourier transforms. A more extensive collection of Fourier transforms can be found in [**89**].

† Since this represents a complex change of variables, a fully rigorous justification of this step requires the use of complex integration.

Concise Table of Fourier Transforms

$f(x)$	$\widehat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\dfrac{\pi}{2}}\,\delta(k) - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,k}$
$\operatorname{sign} x$	$-\,\mathrm{i}\,\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin a\,k}{k}$
$e^{-\,a\,x}\,\sigma(x)$	$\dfrac{1}{\sqrt{2\pi}\,(a + \mathrm{i}\,k)}$
$e^{a\,x}\,(1 - \sigma(x))$	$\dfrac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)}$
$e^{-\,a\,\vert x\vert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{k^2 + a^2}$
$e^{-\,a\,x^2}$	$\dfrac{e^{-\,k^2/(4\,a)}}{\sqrt{2\,a}}$
$\tan^{-1} x$	$\dfrac{\pi^{3/2}}{\sqrt{2}}\,\delta(k) \;-\; \mathrm{i}\,\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-\,\vert k\vert}}{k}$
$f(c\,x + d)$	$\dfrac{e^{\,\mathrm{i}\,k\,d/c}}{\vert c\vert}\,\widehat{f}\left(\dfrac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	$f(-k)$
$f'(x)$	$\mathrm{i}\,k\,\widehat{f}(k)$
$x\,f(x)$	$\mathrm{i}\,\widehat{f}\,'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k)$

Note: The parameters a, c, d are real, with $a > 0$ and $c \neq 0$.

8.2. Derivatives and Integrals.

One of the most significant features of the Fourier transform is that it converts calculus into algebra! More specifically, the two basic operations in calculus — differentiation and integration of functions — are realized as algebraic operations on their Fourier transforms. (The downside is that algebraic operations become more complicated in the frequency domain.)

Differentiation

Let us begin with derivatives. If we differentiate† the basic inverse Fourier transform formula

$$f(x) \ \sim \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,k\,x}\, dk.$$

with respect to x, we obtain

$$f'(x) \ \sim \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{i}\,k\,\widehat{f}(k)\, e^{\,\mathrm{i}\,k\,x}\, dk. \tag{8.40}$$

The resulting integral is itself in the form of an inverse Fourier transform, namely of $\mathrm{i}\,k\,\widehat{f}(k)$ which immediately implies the following key result.

Proposition 8.7. *The Fourier transform of the derivative $f'(x)$ of a function is obtained by multiplication of its Fourier transform by $\mathrm{i}\,k$:*

$$\mathcal{F}[\,f'(x)\,] = \mathrm{i}\,k\,\widehat{f}(k). \tag{8.41}$$

Similarly, the Fourier transform of the product function $x\,f(x)$ is obtained by differentiating the Fourier transform of $f(x)$:

$$\mathcal{F}[\,x\,f(x)\,] = \mathrm{i}\,\frac{d\widehat{f}}{dk}\,. \tag{8.42}$$

The second statement follows easily from the first via the Symmetry Principle of Theorem 8.3. While the result is stated for ordinary functions, as noted earlier, the Fourier transform — just like Fourier series — is entirely compatible with the calculus of generalized functions.

Example 8.8. The derivative of the even exponential pulse $f_e(x) = e^{-a\,|\,x\,|}$ is a multiple of the odd exponential pulse $f_o(x) = (\operatorname{sign} x)\, e^{-a\,|\,x\,|}$:

$$f_e'(x) = -\,a\,(\operatorname{sign} x)\, e^{-a\,|\,x\,|} = -\,a f_o(x).$$

Proposition 8.7 says that their Fourier transforms are related by

$$\mathrm{i}\,k\,\widehat{f}_e(k) = \mathrm{i}\,\sqrt{\frac{2}{\pi}}\;\frac{k\,a}{k^2+a^2} = -\,a\,\widehat{f}_o(k),$$

† We are assuming the integrand is sufficiently nice so that we can bring the derivative under the integral sign; see [**39**, **122**] for a fully rigorous justification.

as previously noted in (8.21, 24). On the other hand, the odd exponential pulse has a jump discontinuity of magnitude 2 at $x = 0$, and so its derivative contains a delta function:

$$f_o'(x) = -\,a\,e^{-a\,|\,x\,|} + 2\,\delta(x) = -\,a\,f_e(x) + 2\,\delta(x).$$

This is reflected in the relation between their Fourier transforms. If we multiply (8.24) by $\mathrm{i}\,k$ we obtain

$$\mathrm{i}\,k\,\widehat{f}_o(k) = \sqrt{\frac{2}{\pi}}\,\frac{k^2}{k^2+a^2} = \sqrt{\frac{2}{\pi}} \;-\; \sqrt{\frac{2}{\pi}}\,\frac{a^2}{k^2+a^2} = 2\,\widehat{\delta}(k) - a\,\widehat{f}_e(k).$$

Higher order derivatives are handled by iterating the first order formula (8.41).

Corollary 8.9. *The Fourier transform of* $f^{(n)}(x)$ *is* $(\,\mathrm{i}\,k)^n\,\widehat{f}(k)$.

This result has an important consequence: the smoothness of the function $f(x)$ is manifested in the rate of decay of its Fourier transform $\widehat{f}(k)$. We already noted that the Fourier transform of a (nice) function must decay to zero at large frequencies: $\widehat{f}(k) \to 0$ as $|\,k\,| \to \infty$. (This result can be viewed as the Fourier transform version of the Riemann–Lebesgue Lemma 3.46.) If the n^{th} derivative $f^{(n)}(x)$ is also a reasonable function, then its Fourier transform $\widehat{f^{(n)}}(k) = (\,\mathrm{i}\,k)^n\,\widehat{f}(k)$ must go to zero as $|\,k\,| \to \infty$. This requires that $\widehat{f}(k)$ go to zero more rapidly than $|\,k\,|^{-n}$. Thus, the smoother $f(x)$, the more rapid the decay of its Fourier transform. As a general rule of thumb, local features of $f(x)$, such as smoothness, are manifested by global features of $\widehat{f}(k)$, such as the rate of decay for large $|\,k\,|$. The Symmetry Principle implies that reverse is also true: global features of $f(x)$ correspond to local features of $\widehat{f}(k)$. For instance, the degree of smoothness of $\widehat{f}(k)$ governs the rate decay of $f(x)$ as $x \to \pm\infty$. This local-global duality is one of the major themes of Fourier theory.

Integration

Integration is the inverse operation to differentiation, and so should correspond to division by $\mathrm{i}\,k$ in frequency space. As with Fourier series, this is not completely correct; there is an extra constant involved, which contributes an additional delta function.

Proposition 8.10. *If* $f(x)$ *has Fourier transform* $\widehat{f}(k)$, *then the Fourier transform of its integral* $g(x) = \int_{-\infty}^{x} f(y)\,dy$ *is*

$$\widehat{g}(k) = -\,\frac{\mathrm{i}}{k}\,\widehat{f}(k) + \pi\,\widehat{f}(0)\,\delta(k). \tag{8.43}$$

Proof: First notice that

$$\lim_{x \to -\infty} g(x) = 0, \qquad\qquad \lim_{x \to +\infty} g(x) = \int_{-\infty}^{\infty} f(x)\,dx = \sqrt{2\pi}\,\widehat{f}(0).$$

Therefore, subtracting a suitable multiple of the step function from the integral, the resulting function

$$h(x) = g(x) - \sqrt{2\pi}\,\widehat{f}(0)\,\sigma(x)$$

decays to 0 at both $\pm\infty$. Consulting our table of Fourier transforms, we find

$$\widehat{h}(k) = \widehat{g}(k) - \pi\,\widehat{f}(0)\,\delta(k) + \frac{\mathrm{i}}{k}\,\widehat{f}(0)\,. \tag{8.44}$$

On the other hand,

$$h'(x) = f(x) - \sqrt{2\pi}\,\widehat{f}(0)\,\delta(x).$$

Since $h(x) \to 0$ as $|\,x\,| \to \infty$, we can apply our differentiation rule (8.41), and conclude that

$$\mathrm{i}\,k\,\widehat{h}(k) = \widehat{f}(k) - \widehat{f}(0). \tag{8.45}$$

Combining (8.44) and (8.45) establishes the desired formula (8.43). *Q.E.D.*

Example 8.11. The Fourier transform of the inverse tangent function

$$f(x) = \tan^{-1} x = \int_0^x \frac{dy}{1+y^2} = \int_{-\infty}^x \frac{dy}{1+y^2} \;-\; \frac{\pi}{2}$$

can be computed by combining Proposition 8.10 with (8.28, 34):

$$\widehat{f}(k) = \left(-\frac{\mathrm{i}}{k}\sqrt{\frac{\pi}{2}}\,\frac{e^{-|\,k\,|}}{k} \;+\; \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k) \right) \;-\; \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k) = -\,\mathrm{i}\sqrt{\frac{\pi}{2}}\,\frac{e^{-|\,k\,|}}{k}\,.$$

The singularity at $k = 0$ reflects the lack of decay of the inverse tangent as $|\,x\,| \to \infty$.

8.3. Green's Functions and Convolution.

The fact that the Fourier transform converts differentiation in the physical domain into multiplication in the frequency domain is one of its most compelling features. A particularly important consequence is that it effectively transforms differential equations into algebraic equations, and thereby facilitates their solution by elementary algebra. One begins by applying the Fourier transform to both sides of the differential equation under consideration. Solving the resulting algebraic equation will produce a formula for the Fourier transform of the desired solution, which can then be immediately reconstructed via the inverse Fourier transform. In the following chapter, we will use these techniques to solve partial differential equations.

Solution of Boundary Value Problems

The Fourier transform is particularly well adapted to boundary value problems on the entire real line. In place of the boundary conditions used on finite intervals, we look for solutions that decay to zero sufficiently rapidly as $|\,x\,| \to \infty$ — in order that their Fourier transform be well-defined (in the context of ordinary functions). In quantum mechanics, [**72**, **78**], these solutions are known as the *bound states*, and correspond to subatomic particles that are trapped or localized in a region of space. For example, the electrons in an atom are bound states localized by the electrostatic attraction of the nucleus.

As a specific example, consider the boundary value problem

$$-\frac{d^2u}{dx^2} + \omega^2\,u = h(x), \qquad -\infty < x < \infty, \tag{8.46}$$

where $\omega > 0$ is a positive constant. The boundary conditions require that the solution decay: $u(x) \to 0$, as $|x| \to \infty$. We will solve this problem by applying the Fourier transform to both sides of the differential equation. Taking Corollary 8.9 into account, the result is the linear algebraic equation

$$k^2\,\widehat{u}(k) + \omega^2\,\widehat{u}(k) = \widehat{h}(k)$$

relating the Fourier transforms of u and h. Unlike the differential equation, the transformed equation can be immediately solved for

$$\widehat{u}(k) = \frac{\widehat{h}(k)}{k^2 + \omega^2}\,. \tag{8.47}$$

Therefore, we can reconstruct the solution by applying the inverse Fourier transform formula (8.9):

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\widehat{h}(k)\,e^{\,i\,k\,x}}{k^2 + \omega^2}\,dk. \tag{8.48}$$

For example, if the forcing function is an even exponential pulse,

$$h(x) = e^{-|x|} \qquad \text{with} \qquad \widehat{h}(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{k^2+1}\,,$$

then (8.48) writes the solution as a Fourier integral:

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{\,i\,k\,x}}{(k^2+\omega^2)(k^2+1)}\,dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos k\,x}{(k^2+\omega^2)(k^2+1)}\,dk\,,$$

noting that the imaginary part of the complex integral vanishes because the integrand is an odd function. (Indeed, if the forcing function is real, the solution must also be real.) The Fourier integral can be explicitly evaluated by using partial fractions to rewrite

$$\widehat{u}(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{(k^2+\omega^2)(k^2+1)} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\omega^2-1}\left(\frac{1}{k^2+1} - \frac{1}{k^2+\omega^2}\right), \qquad \omega^2 \neq 1.$$

Thus, according to our Fourier Transform Table, the solution to this boundary value problem is

$$u(x) = \frac{e^{-|x|} - \dfrac{1}{\omega}\,e^{-\omega\,|x|}}{\omega^2 - 1} \qquad \text{when} \qquad \omega^2 \neq 1. \tag{8.49}$$

The reader may wish to verify that this function is indeed a solution, meaning that it is twice continuously differentiable (which is not so immediately apparent from the formula), decays to 0 as $|x| \to \infty$, and satisfies the differential equation everywhere. The "resonant" case $\omega^2 = 1$ is left to Exercise ■.

Remark: The method of partial fractions that you learned in first year calculus is often an effective tool for evaluating (inverse) Fourier transforms of such rational functions.

A particularly important case is when the forcing function

$$h(x) = \delta_\xi(x) = \delta(x - \xi)$$

represents a unit impulse concentrated at $x = \xi$. The resulting solution is the Green's function $G(x;\xi)$ for the boundary value problem. According to (8.47), its Fourier transform with respect to x is

$$\widehat{G}(k;\xi) = \frac{1}{\sqrt{2\pi}} \; \frac{e^{-\,\mathrm{i}\,k\,\xi}}{k^2 + \omega^2}\,,$$

which is the product of an exponential factor $e^{-\,\mathrm{i}\,k\,\xi}$, representing the Fourier transform of $\delta_\xi(x)$, times a multiple of the Fourier transform of the even exponential pulse $e^{-\,\omega\,|\,x\,|}$. We apply the Shift Theorem 8.4, and conclude that the Green's function for this boundary value problem is an exponential pulse centered at ξ, namely

$$G(x;\xi) = \frac{1}{2\,\omega}\, e^{-\,\omega\,|\,x-\xi\,|} = g(x-\xi), \qquad \text{where} \qquad g(x) = G(x;0) = \frac{1}{2\,\omega}\, e^{-\,\omega\,|\,x\,|}. \tag{8.50}$$

Observe that, as with other self-adjoint boundary value problems, the Green's function is symmetric under interchange of x and ξ, so $G(x;\xi) = G(\xi;x)$. As a function of x, it satisfies the homogeneous differential equation $-\,u'' + \omega^2\, u = 0$, except at the point $x = \xi$ where its derivative has a jump discontinuity of unit magnitude. It also decays as $|\,x\,| \to \infty$, as required by the boundary conditions. The fact that $G(x;\xi) = g(x-\xi)$ only depends on the difference, $x - \xi$, is a consequence of the translation invariance of the boundary value problem. The superposition principle based on the Green's function tells us that the solution to the inhomogeneous boundary value problem (8.46) under a general forcing can be represented in the integral form

$$u(x) = \int_{-\infty}^{\infty} G(x;\xi)\, h(\xi)\, d\xi = \int_{-\infty}^{\infty} g(x-\xi)\, h(\xi)\, d\xi = \frac{1}{2\omega} \int_{-\infty}^{\infty} e^{-\omega|\,x-\xi\,|}\, h(\xi)\, d\xi. \tag{8.51}$$

The reader may enjoy recovering the particular exponential solution (8.49) from this integral formula.

Convolution

In our solution to the boundary value problem (8.46), we ended up deriving a formula for its Fourier transform (8.47) as the product of two known Fourier transforms. The final Green's function formula (8.51), obtained by applying the inverse Fourier transform, is indicative of a general property, in that it is given by a *convolution product*.

Definition 8.12. The *convolution* of scalar functions $f(x)$ and $g(x)$ is the scalar function $h = f * g$ defined by the formula

$$h(x) = f * g(x) = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi. \tag{8.52}$$

We list the basic properties of the convolution product, leaving their verification as exercises for the reader. All of these assume that the implied convolution integrals converge.

(a) *Symmetry*:	$f * g = g * f,$	
(b) *Bilinearity*:	$\begin{cases} f * (a\,g + b\,h) = a\,(f * g) + b\,(f * h), \\ (a\,f + b\,g) * h = a\,(f * h) + b\,(g * h), \end{cases}$	$a, b \in \mathbb{C},$
(c) *Associativity*:	$f * (g * h) = (f * g) * h,$	
(d) *Zero function*:	$f * 0 = 0,$	
(e) *Delta function*:	$f * \delta = f.$	

One tricky feature is that the constant function 1 is *not* a unit for the convolution product; indeed,

$$f * 1 = 1 * f = \int_{-\infty}^{\infty} f(\xi)\, d\xi$$

is a constant function, namely the total integral of f, and not the original function $f(x)$. In fact, according to the final property, the delta function plays the role of the "convolution unit":

$$f * \delta(x) = \int_{-\infty}^{\infty} f(x - \xi)\, \delta(\xi)\, d\xi = f(x).$$

In particular, our solution (8.50) has the form of a convolution product between an even exponential pulse $g(x) = (2\,\omega)^{-1}\, e^{-\omega |\, x\,|}$ and the forcing function:

$$u(x) = g * h(x).$$

On the other hand, its Fourier transform (8.47) is, up to a factor, the ordinary multiplicative product

$$\widehat{u}(k) = \sqrt{2\pi}\; \widehat{g}(k)\, \widehat{h}(k)$$

of the Fourier transforms of g and h. In fact, this is a general property of the Fourier transform: convolution in the physical domain corresponds to multiplication in the frequency domain, and conversely.

Theorem 8.13. *The Fourier transform of the convolution $h(x) = f * g(x)$ of two functions is a multiple of the product of their Fourier transforms*:

$$\widehat{h}(k) = \sqrt{2\pi}\; \widehat{f}(k)\, \widehat{g}(k). \tag{8.53}$$

Vice versa, the Fourier transform of their product $h(x) = f(x)\, g(x)$ is, up to multiple, the convolution of their Fourier transforms:

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}}\, \widehat{f} * \widehat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k - \kappa)\, \widehat{g}(\kappa)\, d\kappa. \tag{8.54}$$

Proof: Combining the definition of the Fourier transform with the convolution formula (8.52), we find

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\, e^{-\,\mathrm{i}\,k\,x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - \xi)\, g(\xi)\, e^{-\,\mathrm{i}\,k\,x}\, dx\, d\xi.$$

Applying the change of variables $\eta = x - \xi$ in the inner integral produces

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta)\, g(\xi)\, e^{-\mathrm{i}\, k(\xi+\eta)}\, d\xi\, d\eta$$

$$= \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\, e^{-\mathrm{i}\, k\eta}\, d\eta \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi)\, e^{-\mathrm{i}\, k\xi}\, d\xi \right) = \sqrt{2\pi}\, \widehat{f}(k)\, \widehat{g}(k),$$

proving (8.53). The second formula can be proved in a similar fashion, or by simply noting that it follows directly from the Symmetry Principle of Theorem 8.3. *Q.E.D.*

Example 8.14. We already know, (8.29), that the Fourier transform of

$$f(x) = \frac{\sin x}{x}$$

is the box function

$$\widehat{f}(k) = \sqrt{\frac{\pi}{2}}\, \big[\, \sigma(k+1) - \sigma(k-1)\, \big] = \begin{cases} \sqrt{\dfrac{\pi}{2}}\,, & -1 < k < 1, \\ 0, & |\, k\,| > 1. \end{cases}$$

We also know that the Fourier transform of

$$g(x) = \frac{1}{x} \qquad \text{is} \qquad \widehat{g}(k) = -\,\mathrm{i}\, \sqrt{\frac{\pi}{2}}\, \operatorname{sign} k.$$

Therefore, the Fourier transform of their product

$$h(x) = f(x)\, g(x) = \frac{\sin x}{x^2}$$

can be obtained by convolution:

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}}\, \widehat{f} * \widehat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\kappa)\, \widehat{g}(k-\kappa)\, d\kappa$$

$$= -\,\mathrm{i}\, \sqrt{\frac{\pi}{8}} \int_{-1}^{1} \operatorname{sign}(k-\kappa)\, d\kappa = \begin{cases} \mathrm{i}\, \sqrt{\dfrac{\pi}{2}} & k < -1, \\ -\,\mathrm{i}\, \sqrt{\dfrac{\pi}{2}}\, k, & -1 < k < 1, \\ -\,\mathrm{i}\, \sqrt{\dfrac{\pi}{2}} & k > 1. \end{cases}$$

A graph of $\widehat{h}(k)$ appears in Figure 8.3.

8.4. The Fourier Transform on Hilbert Space.

While we do not possess all the analytical tools to embark on a fully rigorous treatment of the mathematical theory underlying the Fourier transform, it is worth outlining a few of the more important features. We have already noted that the Fourier transform, when defined, is a linear map, taking functions $f(x)$ on physical space to functions $\widehat{f}(k)$ on

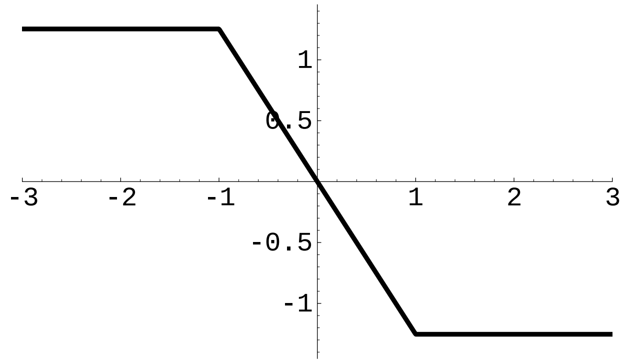


Figure 8.3. The Fourier transform of $\dfrac{\sin x}{x^2}$.

frequency space. A critical question is to precisely which function space should the theory be applied. Not every function admits a Fourier transform in the classical sense† — the Fourier integral (8.6) is required to converge, and this places restrictions on the function and its asymptotics at large distances.

It turns out the proper setting for the rigorous theory is the *Hilbert space* of complex-valued square-integrable functions — the same infinite-dimensional vector space that lies at the heart of modern quantum mechanics. In Section 3.5, we already introduced the Hilbert space $\mathrm{L}^2[a,b]$ on a finite interval; here we adapt Definition 3.34 to the entire real line. Thus, the Hilbert space $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R})$ is the infinite-dimensional vector space consisting of all complex-valued functions $f(x)$ which are defined for all $x \in \mathbb{R}$ and have finite L^2 norm:

$$\| f \|^2 = \int_{-\infty}^{\infty} | f(x) |^2 \, dx < \infty. \tag{8.55}$$

For example, any piecewise continuous function that satisfies the decay criterion

$$| f(x) | \leq \frac{M}{| x |^{1/2+\delta}}, \qquad \text{for all sufficiently large} \qquad | x | \gg 0, \tag{8.56}$$

for some $M > 0$ and $\delta > 0$, belongs to L^2. However, as in Section 3.5, Hilbert space contains many more functions, and the precise definitions and identification of its elements is quite subtle. On the other hand, most non-decaying functions do not belong to L^2, including the constant function $f(x) \equiv 1$ as well as all oscillatory complex exponentials, $e^{\,\mathrm{i}\,kx}$ for $k \in \mathbb{R}$.

The Hermitian inner product on the complex Hilbert space L^2 is prescribed in the usual manner,

$$\langle f \,; g \rangle = \int_{-\infty}^{\infty} f(x) \, \overline{g(x)} \, dx, \tag{8.57}$$

so that $\| f \|^2 = \langle f \,; f \rangle$. The Cauchy–Schwarz inequality

$$| \langle f \,; g \rangle | \leq \| f \| \, \| g \| \tag{8.58}$$

† We leave aside the more advanced issues involving generalized functions.

ensures that the inner product integral is finite whenever $f, g \in \mathrm{L}^2$. Observe that the Fourier transform (8.6) can be regarded as a multiple of the inner product of the function $f(x)$ with the complex exponential functions:

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-\,\mathrm{i}\,k\,x}\, dx = \frac{1}{\sqrt{2\pi}} \,\langle\, f(x) \,;\, e^{\,\mathrm{i}\,k\,x} \,\rangle. \tag{8.59}$$

However, when interpreting this formula, one must bear in mind that the exponentials are *not* themselves elements of L^2.

Let us state the fundamental result governing the effect of the Fourier transform on functions in Hilbert space. It can be regarded as a direct analog of the Pointwise Convergence Theorem 3.8 for Fourier series.

Theorem 8.15. *If $f(x) \in \mathrm{L}^2$ is square-integrable, then its Fourier transform $\widehat{f}(k) \in \mathrm{L}^2$ is a well-defined, square-integrable function of the frequency variable k. If $f(x)$ is continuously differentiable at a point x, then the inverse Fourier transform integral* (8.9) *equals its value $f(x)$. More generally, if the right and left hand limits $f(x^-)$, $f(x^+)$, $f'(x^-)$, $f'(x^+)$ exist, then the inverse Fourier transform integral converges to the average value $\frac{1}{2}\left[\, f(x^-) + f(x^+) \,\right]$.*

Thus, the Fourier transform $\widehat{f} = \mathcal{F}[\,f\,]$ defines a linear transformation from L^2 functions of x to L^2 functions of k. In fact, the Fourier transform preserves inner products. This important result is known as *Parseval's formula*, whose Fourier series counterpart appeared in (3.122).

Theorem 8.16. *If $\widehat{f}(k) = \mathcal{F}[\,f(x)\,]$ and $\widehat{g}(k) = \mathcal{F}[\,g(x)\,]$, then $\langle\, f \,;\, g \,\rangle = \langle\, \widehat{f} \,;\, \widehat{g} \,\rangle$, i.e.,*

$$\int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx = \int_{-\infty}^{\infty} \widehat{f}(k)\, \overline{\widehat{g}(k)}\, dk. \tag{8.60}$$

Proof: Let us sketch a formal proof that serves to motivate why this result is valid. We use the definition (8.6) of the Fourier transform to evaluate

$$\begin{aligned} \int_{-\infty}^{\infty} \widehat{f}(k)\, \overline{\widehat{g}(k)}\, dk &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-\,\mathrm{i}\,k\,x}\, dx \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{g(y)}\, e^{+\,\mathrm{i}\,k\,y}\, dy \right) dk \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \overline{g(y)} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\,\mathrm{i}\,k(x-y)}\, dk \right) dx\, dy. \end{aligned}$$

Now according to (8.38), the inner k integral can be replaced by the delta function $\delta(x-y)$, and hence

$$\int_{-\infty}^{\infty} \widehat{f}(k)\, \overline{\widehat{g}(k)}\, dk = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \overline{g(y)}\, \delta(x-y)\, dx\, dy = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx.$$

This completes our "proof"; see [**39**, **74**, **122**] for a rigorous version. *Q.E.D.*

In particular, orthogonal functions, satisfying $\langle f\,;g \rangle = 0$, will have orthogonal Fourier transforms, $\langle \widehat{f}\,;\widehat{g} \rangle = 0$. Choosing $f = g$ in Parseval's formula (8.60) results in the *Plancherel formula*

$$\| f \|^2 = \| \widehat{f} \|^2, \qquad \text{or, explicitly,} \qquad \int_{-\infty}^{\infty} |\, f(x)\,|^2\, dx = \int_{-\infty}^{\infty} |\, \widehat{f}(k)\,|^2\, dk. \tag{8.61}$$

Thus, the Fourier transform $\mathcal{F}\colon \mathrm{L}^2 \to \mathrm{L}^2$ defines a norm-preserving or *unitary* linear transformation on Hilbert space, mapping L^2 functions of the physical variable x to L^2 functions of the frequency variable k.

Quantum Mechanics and the Uncertainty Principle

In its popularized form, the Heisenberg Uncertainty Principle is a by now familiar philosophical concept. It was first formulated in the 1920's by the German physicist Werner Heisenberg, one of the founders of modern quantum mechanics, and states that, in a physical system, certain quantities cannot be simultaneously measured with complete accuracy. For instance, the more precisely one measures the position of a particle, the less accuracy there will be in the measurement of its momentum; vice versa, the greater the accuracy in the momentum, the less certainty in its position. A similar uncertainty couples energy and time. Experimental verification of the uncertainty principle can be found even in fairly simple situations. Consider a light beam passing through a small hole. The position of the photons is constrained by the hole; the effect of their momenta is in the pattern of light diffused on a screen placed beyond the hole. The smaller the hole, the more constrained the position, and the wider the image on the screen, meaning the less certainty there is in the observed momentum.

This is not the place to discuss the philosophical and experimental consequences of Heisenberg's principle. What we will show is that the Uncertainty Principle is, in fact, a mathematical property of the Fourier transform! In quantum theory, each of the paired quantities, e.g., position and momentum, are interrelated by the Fourier transform. Indeed, Proposition 8.7 says that the Fourier transform of the differentiation operator representing momentum is a multiplication operator representing position and vice versa. This Fourier transform-based duality between position and momentum, that is, between multiplication and differentiation, lies at the heart of the Uncertainty Principle.

In quantum mechanics, the wave functions of a quantum system are characterized as the elements of unit norm, $\| \varphi \| = 1$, belonging to the underlying state space, which, in a one-dimensional model of a single particle, is the Hilbert space $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R})$ consisting of square integrable, complex valued functions of x. As we already noted in Section 3.5, the squared modulus of the wave function, $|\, \varphi(x)\,|^2$, represents the probability density of the particle being found at position x. Consequently, the *mean* or *expected value* of any function $f(x)$ of the position variable is given by its integral against the system's probability density, and denoted by

$$\langle\, f(x) \,\rangle = \int_{-\infty}^{\infty} f(x)\, |\, \varphi(x)\,|^2\, dx. \tag{8.62}$$

In particular,

$$\langle\, x\,\rangle = \int_{-\infty}^{\infty} x\,|\,\varphi(x)\,|^2\,dx \tag{8.63}$$

is the expected measured position of the particle, while Δx, defined by

$$(\Delta x)^2 = \langle\, \big(\,x - \langle\, x\,\rangle\,\big)^2\,\rangle = \langle\, x^2\,\rangle - \langle\, x\,\rangle^2 \tag{8.64}$$

is the *variance*, that is, the statistical deviation of the particle's measured position from the mean. We note that the next-to-last term equals

$$\langle\, x^2\,\rangle = \int_{-\infty}^{\infty} x^2\,|\,\varphi(x)\,|^2\,dx = \|\, x\,\varphi(x)\,\|^2. \tag{8.65}$$

On the other hand, the momentum variable p is related to the Fourier transform frequency via the *de Broglie relation* $p = \hbar\, k$, where

$$\hbar = \frac{h}{2\pi} \approx 1.055\times 10^{-34} \ \ \text{Joule seconds} \tag{8.66}$$

is *Planck's constant*, whose value governs the quantization of physical quantities. Therefore, the mean or expected value of any function of momentum $g(p)$ is given by its integral against the squared modulus of the Fourier transformed wave function:

$$\langle\, g(p)\,\rangle = \int_{-\infty}^{\infty} g(\hbar\, k)\,|\,\widehat{\varphi}(k)\,|^2\,dk. \tag{8.67}$$

In particular, the mean of the momentum measurements of the particle is

$$\langle\, p\,\rangle = \hbar \int_{-\infty}^{\infty} k\,|\,\widehat{\varphi}(k)\,|^2\,dk = -\,\mathrm{i}\,\hbar \int_{-\infty}^{\infty} \varphi'(x)\,\overline{\varphi(x)}\,dx = -\,\mathrm{i}\,\hbar\,\langle\,\varphi'\,;\varphi\,\rangle, \tag{8.68}$$

where we used Parseval's formula (8.60) to convert to an integral over position, and (8.41) to infer that $k\,\widehat{\varphi}(k)$ is the Fourier transform of $-\,\mathrm{i}\,\varphi'(x)$. Similarly,

$$(\Delta p)^2 = \langle\, \big(\,p - \langle\, p\,\rangle\,\big)^2\,\rangle = \langle\, p^2\,\rangle - \langle\, p\,\rangle^2 \tag{8.69}$$

is the squared *variance* of the momentum, where, by Plancherel's formula (8.61) and (8.41),

$$\begin{aligned}\langle\, p^2\,\rangle = \hbar^2 \int_{-\infty}^{\infty} k^2\,|\,\widehat{\varphi}(k)\,|^2\,dk &= \hbar^2 \int_{-\infty}^{\infty} |\,\mathrm{i}\,k\,\widehat{\varphi}(k)\,|^2\,dk \\ &= \hbar^2 \int_{-\infty}^{\infty} |\,\varphi'(x)\,|^2\,dx = \hbar^2\,\|\,\varphi'(x)\,\|^2.\end{aligned} \tag{8.70}$$

With this interpretation, the Uncertainty Principle for position and momentum measurements can be stated.

Theorem 8.17. *If $\varphi(x)$ is a wave function, so $\|\,\varphi\,\| = 1$, then the observed variances in position and momentum satisfy the inequality*

$$\Delta x \,\Delta p \geq \tfrac{1}{2}\,\hbar. \tag{8.71}$$

Now, the smaller the variance of a quantity such as position or momentum, the more accurate will be its measurement. Thus, the Heisenberg inequality (8.71) effectively quantifies the statement that the more accurately we are able to measure the momentum p, the less accurate will be any measurement of its position x, and vice versa. For more details, along with physical and experimental consequences, you should consult an introductory text on mathematical quantum mechanics, e.g., [**72**, **78**].

Proof: For any value of the real parameter t,

$$\begin{aligned} 0 &\leq \|\, t\, x\, \varphi(x) + \varphi'(x)\,\|^2 \\ &= t^2\,\|\, x\,\varphi(x)\,\|^2 + t\,\big[\,\langle\, \varphi'(x)\,;x\,\varphi(x)\,\rangle + \langle\, x\,\varphi(x)\,;\varphi'(x)\,\rangle\,\big] + \|\,\varphi'(x)\,\|^2. \end{aligned} \tag{8.72}$$

The middle term in the final expression can be evaluated as follows:

$$\begin{aligned} \langle\, \varphi'(x)\,;x\,\varphi(x)\,\rangle + \langle\, x\,\varphi(x)\,;\varphi'(x)\,\rangle &= \int_{-\infty}^{\infty} \big[\, x\,\varphi'(x)\,\overline{\varphi(x)} + x\,\varphi(x)\,\overline{\varphi'(x)}\,\big]\, dx \\ = \int_{-\infty}^{\infty} x\,\frac{d}{dx}\,|\,\varphi(x)\,|^2\,dx &= -\int_{-\infty}^{\infty} |\,\varphi(x)\,|^2\,dx = -1, \end{aligned}$$

via an integration by parts, noting that the boundary terms vanish provided $\varphi(x)$ satisfies the L^2 decay criterion (8.56). Thus, in view of (8.65) and (8.70), the inequality in (8.72) reads

$$\langle\, x^2\,\rangle\, t^2 - t + \frac{\langle\, p^2\,\rangle}{\hbar^2} \geq 0 \qquad \text{for all} \qquad t \in \mathbb{R}.$$

The minimum value of the left hand side occurs at $t_\star = 1/(2\,\langle\, x^2\,\rangle)$, where its value is

$$\frac{\langle\, p^2\,\rangle}{\hbar^2} - \frac{1}{4\,\langle\, x^2\,\rangle} \geq 0 \qquad \text{which implies} \qquad \langle\, x^2\,\rangle\,\langle\, p^2\,\rangle \geq \tfrac{1}{4}\,\hbar^2.$$

To obtain the Uncertainty Relation (8.71), one performs the selfsame calculation, but with $x - \langle\, x\,\rangle$ replacing x and $p - \langle\, p\,\rangle$ replacing p. The result is

$$\big\langle\, (x - \langle\, x\,\rangle)^2\,\big\rangle\, t^2 - t + \frac{\big\langle\, (p - \langle\, p\,\rangle)^2\,\big\rangle}{\hbar^2} = (\Delta x)^2\, t^2 - t + \frac{(\Delta p)^2}{\hbar^2} \geq 0. \tag{8.73}$$

Substituting $t = 1/(2\,(\Delta x)^2)$ produces the Heisenberg inequality (8.71). *Q.E.D.*

