Introduction to Partial Differential Equations

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Preface

Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering. For example, the system of partial differential equations known as Maxwell's equations can be written on the back of a post card, yet from these equations one can derive the entire theory of electricity and magnetism, including light.

Our goal here is to develop the most basic ideas from the theory of partial differential equations, and apply them to the simplest models arising from physics. In particular, we will present some of the elegant mathematics that can be used to describe the vibrating circular membrane. We will see that the frequencies of a circular drum are essentially the eigenvalues from an eigenvector-eigenvalue problem for Bessel's equation, an ordinary differential equation which can be solved quite nicely using the technique of power series expansions. Thus we start our presentation with a review of power series, which the student should have seen in a previous calculus course.

It is not easy to master the theory of partial differential equations. Unlike the theory of ordinary differential equations, which relies on the "fundamental existence and uniqueness theorem," there is no single theorem which is central to the subject. Instead, there are separate theories used for each of the major types of partial differential equations that commonly arise.

However, there are several basic skills which are essential for studying all types of partial differential equations. Before reading these notes, students should understand how to solve the simplest ordinary differential equations, such as the equation of exponential growth $dy/dx = ky$ and the equation of simple harmonic motion $d^2y/dx^2 + \omega y = 0$, and how these equations arise in modeling population growth and the motion of a weight attached to the ceiling by means of a spring. It is remarkable how frequently these basic equations arise in applications. Students should also understand how to solve first-order linear systems of differential equations with constant coefficients in an arbitrary number of unknowns using vectors and matrices with real or complex entries. (This topic will be reviewed in the second chapter.) Familiarity is also needed with the basics of vector calculus, including the gradient, divergence and curl, and the integral theorems which relate them to each other. Finally, one needs ability to carry out lengthy calculations with confidence. Needless to say, all of these skills are necessary for a thorough understanding of the mathematical language that is an essential foundation for the sciences and engineering.

Moreover, the subject of partial differential equations should not be studied in isolation, because much intuition comes from a thorough understanding of applications. The individual branches of the subject are concerned with the special types of partial differential equations which which are needed to model diffusion, wave motion, equilibria of membranes and so forth. The behavior of physical systems often suggest theorems which can be proven via rigorous mathematics. (This last point, and the book itself, can be best appreciated by those who have taken a course in rigorous mathematical proof, such as a course in mathematical inquiry, whether at the high school or university level.)

Moreover, the objects modeled make it clear that there should be a constant tension between the discrete and continuous. For example, a vibrating string can be regarded profitably as a continuous object, yet if one looks at a fine enough scale, the string is made up of molecules, suggesting a discrete model with a large number of variables. Moreover, we will see that although a partial differential equation provides an elegant continuous model for a vibrating membrane, the numerical method used to do actual calculations may approximate this continuous model with a discrete mechanical system with a large number of degrees of freedom. The eigenvalue problem for a differential equation thereby becomes approximated by an eigenvalue problem for an $n \times n$ matrix where n is large, thereby providing a link between the techniques studied in linear algebra and those of partial differential equations. The reader should be aware that there are many cases in which a discrete model may actually provide a better description of the phenomenon under study than a continuous one. One should also be aware that probabilistic techniques provide an additional component to model building, alongside the partial differential equations and discrete mechanical systems with many degrees of freedom described in these pages.

There is a major dichotomy that runs through the subject—linear versus nonlinear. It is actually linear partial differential equations for which the technique of linear algebra prove to be so effective. This book is concerned primarly with linear partial differential equations—yet it is the nonlinear partial differential equations that provide the most intriguing questions for research. Nonlinear partial differential equations include the Einstein field equations from general relativity and the Navier-Stokes equations which describe fluid motion. We hope the linear theory presented here will whet the student's appetite for studying the deeper waters of the nonlinear theory.

The author would appreciate comments that may help improve the next version of this short book. He hopes to make a list of corrections available at the web site:

`http://www.math.ucsb.edu/~moore`

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The sections marked with asterisks are less central to the main line of discussion, and may be treated briefly or omitted if time runs short.

Chapter 1

Power Series

1.1 What is a power series?

Functions are often represented efficiently by means of infinite series. Examples we have seen in calculus include the exponential function

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \tag{1.1}$$

as well as the trigonometric functions

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k}$$

and

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}x^{2k+1}.$$

An infinite series of this type is called a power series. To be precise, a *power series* centered at x_0 is an infinite sum of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where the a_n's are constants. In advanced treatments of calculus, these power series representations are often used to define the exponential and trigonometric functions.

Power series can also be used to construct tables of values for these functions. For example, using a calculator or PC with suitable software installed (such as Mathematica), we could calculate

$$1 + 1 + \frac{1}{2!}1^2 = \sum_{n=0}^{2} \frac{1}{n!}1^n = 2.5,$$

$$1 + 1 + \frac{1}{2!}1^2 + \frac{1}{3!}1^3 + \frac{1}{4!}1^4 = \sum_{n=0}^{4} \frac{1}{n!}1^n = 2.70833,$$

$$\sum_{n=0}^{8} \frac{1}{n!}1^n = 2.71806, \qquad \sum_{n=0}^{12} \frac{1}{n!}1^n = 2.71828, \qquad \ldots$$

As the number of terms increases, the sum approaches the familiar value of the exponential function e^x at $x = 1$.

For a power series to be useful, the infinite sum must actually add up to a finite number, as in this example, for at least some values of the variable x. We let s_N denote the sum of the first $(N + 1)$ terms in the power series,

$$s_N = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_N(x - x_0)^N = \sum_{n=0}^{N} a_n(x - x_0)^n,$$

and say that the power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

converges if the finite sum s_N gets closer and closer to some (finite) number as $N \to \infty$.

Let us consider, for example, one of the most important power series of applied mathematics, the *geometric series*

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

In this case we have

$$s_N = 1 + x + x^2 + x^3 + \cdots + x^N, \qquad x s_N = x + x^2 + x^3 + x^4 \cdots + x^{N+1},$$

$$s_N - x s_N = 1 - x^{N+1}, \qquad s_N = \frac{1 - x^{N+1}}{1 - x}.$$

If $|x| < 1$, then x^{N+1} gets smaller and smaller as N approaches infinity, and hence

$$\lim_{N \to \infty} x^{N+1} = 0.$$

Substituting into the expression for s_N, we find that

$$\lim_{N \to \infty} s_N = \frac{1}{1 - x}.$$

Thus if $|x| < 1$, we say that the geometric series converges, and write

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

On the other hand, if $|x| > 1$, then x^{N+1} gets larger and larger as N approaches infinity, so $\lim_{N\to\infty} x^{N+1}$ does not exist as a finite number, and neither does $\lim_{N\to\infty} s_N$. In this case, we say that the geometric series *diverges*. In summary, the geometric series

$$\sum_{n=0}^{\infty} x^n \qquad \text{converges to} \quad \frac{1}{1-x} \qquad \text{when} \quad |x| < 1,$$

and diverges when $|x| > 1$.

This behaviour, convergence for $|x| <$ some number, and divergences for $|x| >$ that number, is typical of power series:

Theorem. *For any power series*

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

there exists R, which is a nonnegative real number or ∞, such that

1. *the power series converges when $|x - x_0| < R$,*
2. *and the power series diverges when $|x - x_0| > R$.*

We call R the *radius of convergence*. A proof of this theorem is given in more advanced courses on real or complex analysis.[1]

We have seen that the geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

has radius of convergence $R = 1$. More generally, if b is a positive constant, the power series

$$1 + \frac{x}{b} + \left(\frac{x}{b}\right)^2 + \left(\frac{x}{b}\right)^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{x}{b}\right)^n \tag{1.2}$$

has radius of convergence b. To see this, we make the substitution $y = x/b$, and the power series becomes $\sum_{n=0}^{\infty} y^n$, which we already know converges for $|y| < 1$ and diverges for $|y| > 1$. But

$$|y| < 1 \quad \Leftrightarrow \quad \left|\frac{x}{b}\right| < 1 \quad \Leftrightarrow \quad |x| < b,$$

$$|y| > 1 \quad \Leftrightarrow \quad \left|\frac{x}{b}\right| > 1 \quad \Leftrightarrow \quad |x| > b.$$

[1]Good references for the theory behind convergence of power series are Edward D. Gaughan, *Introduction to analysis*, Brooks/Cole Publishing Company, Pacific Grove, 1998 and Walter Rudin, *Principles of mathematical analysis*, third edition, McGraw-Hill, New York, 1976.

Thus for $|x| < b$ the power series (1.2) converges to

$$\frac{1}{1-y} = \frac{1}{1-(x/b)} = \frac{b}{b-x},$$

while for $|x| > b$, it diverges.

There is a simple criterion that often enables one to determine the radius of convergence of a power series.

Ratio Test. The radius of convergence of the power series

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

is given by the formula

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

so long as this limit exists.

Let us check that the ratio test gives the right answer for the radius of convergence of the power series (1.2). In this case, we have

$$a_n = \frac{1}{b^n}, \quad \text{so} \quad \frac{|a_n|}{|a_{n+1}|} = \frac{1/b^n}{1/b^{n+1}} = \frac{b^{n+1}}{b^n} = b,$$

and the formula from the ratio test tells us that the radius of convergence is $R = b$, in agreement with our earlier determination.

In the case of the power series for e^x,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

in which $a_n = 1/n!$, we have

$$\frac{|a_n|}{|a_{n+1}|} = \frac{1/n!}{1/(n+1)!} = \frac{(n+1)!}{n!} = n + 1,$$

and hence

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} (n + 1) = \infty,$$

so the radius of convergence is infinity. In this case the power series converges for all x. In fact, we could use the power series expansion for e^x to calculate e^x for any choice of x.

On the other hand, in the case of the power series

$$\sum_{n=0}^{\infty} n! x^n,$$

in which $a_n = n!$, we have

$$\frac{|a_n|}{|a_{n+1}|} = \frac{n!}{(n+1)!} = \frac{1}{n+1}, \qquad R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0.$$

In this case, the radius of convergence is zero, and the power series does not converge for any nonzero x.

The ratio test doesn't always work because the limit may not exist, but sometimes one can use it in conjunction with the

Comparison Test. Suppose that the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

have radius of convergence R_1 and R_2 respectively. If $|a_n| \leq |b_n|$ for all n, then $R_1 \geq R_2$. If $|a_n| \geq |b_n|$ for all n, then $R_1 \leq R_2$.

In short, power series with smaller coefficients have larger radius of convergence.

Consider for example the power series expansion for $\cos x$,

$$1 + 0x - \frac{1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}.$$

In this case the coefficient a_n is zero when n is odd, while $a_n = \pm 1/n!$ when n is even. In either case, we have $|a_n| \leq 1/n!$. Thus we can compare with the power series

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

which represents e^x and has infinite radius of convergence. It follows from the comparison test that the radius of convergence of

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}$$

must be at least large as that of the power series for e^x, and hence must also be infinite.

Power series with positive radius of convergence are so important that there is a special term for describing functions which can be represented by such power series. A function $f(x)$ is said to be *real analytic* at x_0 if there is a power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

about x_0 with positive radius of convergence R such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \text{for} \qquad |x-x_0| < R.$$

For example, the functions e^x is real analytic at any x_0. To see this, we utilize the law of exponents to write $e^x = e^{x_0} e^{x - x_0}$ and apply (1.1) with x replaced by $x - x_0$:

$$e^x = e^{x_0} \sum_{n=0}^{\infty} \frac{1}{n!}(x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad \text{where} \qquad a_n = \frac{e^{x_0}}{n!}.$$

This is a power series expansion of e^x about x_0 with infinite radius of convergence. Similarly, the monomial function $f(x) = x^n$ is real analytic at x_0 because

$$x^n = (x - x_0 + x_0)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} x_0^{n-i} (x - x_0)^i$$

by the binomial theorem, a power series about x_0 in which all but finitely many of the coefficients are zero.

In more advanced courses, one studies criteria under which functions are real analytic. For our purposes, it is sufficient to be aware of the following facts: The sum and product of real analytic functions is real analytic. It follows from this that any polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is analytic at any x_0. The quotient of two polynomials with no common factors, $P(x)/Q(x)$, is analytic at x_0 if and only if x_0 is not a zero of the denominator $Q(x)$. Thus for example, $1/(x-1)$ is analytic whenever $x_0 \neq 1$, but fails to be analytic at $x_0 = 1$.

Exercises:

1.1.1. Use the ratio test to find the radius of convergence of the following power series:

a. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$, b. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1} x^n$,

c. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{n+1} (x-2)^n$, d. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} (x-\pi)^n$,

e. $\displaystyle\sum_{n=0}^{\infty} (7x-14)^n$, f. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} (3x-6)^n$.

1.1.2. Use the comparison test to find an estimate for the radius of convergence of each of the following power series:

a. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$, b. $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k}$,

c. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2k} (x-4)^{2k}$ d. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{2k}} (x-\pi)^{2k}$.

1.1.3. Use the comparison test and the ratio test to find the radius of convergence of the power series

$$\sum_{m=0}^{\infty}(-1)^m \frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m}.$$

1.1.4. Determine the values of x_0 at which the following functions fail to be real analytic:

$$\text{a.} \quad f(x)=\frac{1}{x-4}, \qquad \text{b.} \quad g(x)=\frac{x}{x^2-1},$$

$$\text{c.} \quad h(x)=\frac{4}{x^4-3x^2+2}, \qquad \text{d.} \quad \phi(x)=\frac{1}{x^3-5x^2+6x}$$

1.2 Solving differential equations by means of power series

Our main goal in this chapter is to study how to determine solutions to differential equations by means of power series. As an example, we consider our old friend, the equation of simple harmonic motion

$$\frac{d^2y}{dx^2}+y=0, \tag{1.3}$$

which we have already learned how to solve by other methods. Suppose for the moment that we don't know the general solution and want to find it by means of power series. We could start by assuming that

$$y=a_0+a_1x+a_2x^2+a_3x^3+\cdots=\sum_{n=0}^{\infty}a_nx^n. \tag{1.4}$$

It can be shown that the standard technique for differentiating polynomials term by term also works for power series, so we expect that

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+\cdots=\sum_{n=1}^{\infty}na_nx^{n-1}.$$

(Note that the last summation only goes from 1 to ∞, since the term with $n=0$ drops out of the sum.) Differentiating again yields

$$\frac{d^2y}{dx^2}=2a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+\cdots=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}.$$

We can replace n by $m+2$ in the last summation so that

$$\frac{d^2y}{dx^2}=\sum_{m+2=2}^{\infty}(m+2)(m+2-1)a_{m+2}x^{m+2-2}=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m.$$

The index m is a "dummy variable" in the summation and can be replaced by any other letter. Thus we are free to replace m by n and obtain the formula

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Substitution into equation(1.3) yields

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0,$$

or

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + a_n]x^n = 0.$$

Recall that a polynomial is zero only if all its coefficients are zero. Similarly, a power series can be zero only if all of its coefficients are zero. It follows that

$$(n+2)(n+1)a_{n+2} + a_n = 0,$$

or

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}. \tag{1.5}$$

This is called a *recursion formula* for the coefficients a_n.

The first two coefficients a_0 and a_1 in the power series can be determined from the initial conditions,

$$y(0) = a_0, \qquad \frac{dy}{dx}(0) = a_1.$$

Then the recursion formula can be used to determine the remaining coefficients by the process of induction. Indeed it follows from (1.5) with $n = 0$ that

$$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{1}{2}a_0.$$

Similarly, it follows from (1.5) with $n = 1$ that

$$a_3 = -\frac{a_1}{3\cdot 2} = -\frac{1}{3!}a_1,$$

and with $n = 2$ that

$$a_4 = -\frac{a_2}{4\cdot 3} = \frac{1}{4\cdot 3}\frac{1}{2}a_0 = \frac{1}{4!}a_0.$$

Continuing in this manner, we find that

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0, \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1.$$

Substitution into (1.4) yields

$$
\begin{aligned}
y &= a_0 + a_1 x - \frac{1}{2!}a_0 x^2 - \frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \cdots \\
&= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right).
\end{aligned}
$$

We recognize that the expressions within parentheses are power series expansions of the functions $\sin x$ and $\cos x$, and hence we obtain the familiar expression for the solution to the equation of simple harmonic motion,

$$y = a_0 \cos x + a_1 \sin x.$$

The method we have described—assuming a solution to the differential equation of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and solve for the coefficients a_n—is surprisingly effective, particularly for the class of equations called second-order linear differential equations.

It is proven in books on differential equations that if $P(x)$ and $Q(x)$ are well-behaved functions, then the solutions to the "homogeneous linear differential equation"

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

can be organized into a two-parameter family

$$y = a_0 y_0(x) + a_1 y_1(x),$$

called the *general solution*. Here $y_0(x)$ and $y_1(x)$ are any two nonzero solutions, neither of which is a constant multiple of the other. In the terminology used in linear algebra, we say that they are linearly independent solutions. As a_0 and a_1 range over all constants, y ranges throughout a "linear space" of solutions. We say that $y_0(x)$ and $y_1(x)$ form a *basis* for the space of solutions.

In the special case where the functions $P(x)$ and $Q(x)$ are real analytic, the solutions $y_0(x)$ and $y_1(x)$ will also be real analytic. This is the content of the following theorem, which is proven in more advanced books on differential equations:

Theorem. *If the functions $P(x)$ and $Q(x)$ can be represented by power series*

$$P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

with positive radii of convergence R_1 and R_2 respectively, then any solution $y(x)$ to the linear differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

can be represented by a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

whose radius of convergence is $\geq$ *the smallest of* R_1 *and* R_2.

This theorem is used to justify the solution of many well-known differential equations by means of the power series method.

Example. *Hermite's differential equation* is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2py = 0, \tag{1.6}$$

where p is a parameter. It turns out that this equation is very useful for treating the simple harmonic oscillator in quantum mechanics, but for the moment, we can regard it as merely an example of an equation to which the previous theorem applies. Indeed,

$$P(x) = -2x, \qquad Q(x) = 2p,$$

both functions being polynomials, hence power series about $x_0 = 0$ with infinite radius of convergence.

As in the case of the equation of simple harmonic motion, we write

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We differentiate term by term as before, and obtain

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Once again, we can replace n by $m+2$ in the last summation so that

$$\frac{d^2y}{dx^2} = \sum_{m+2=2}^{\infty} (m+2)(m+2-1) a_{m+2} x^{m+2-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m,$$

and then replace m by n once again, so that

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \tag{1.7}$$

Note that

$$-2x\frac{dy}{dx} = \sum_{n=0}^{\infty} -2n a_n x^n, \tag{1.8}$$

while

$$2py = \sum_{n=0}^{\infty} 2pa_n x^n. \tag{1.9}$$

Adding together (1.7), (1.8) and (1.9), we obtain

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2py = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(-2n+2p)a_n x^n.$$

If y satisfies Hermite's equation, we must have

$$0 = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}(-2n+2p)a_n]x^n.$$

Since the right-hand side is zero for all choices of x, each coefficient must be zero, so

$$(n+2)(n+1)a_{n+2} + (-2n+2p)a_n = 0,$$

and we obtain the *recursion formula* for the coefficients of the power series:

$$a_{n+2} = \frac{2n-2p}{(n+2)(n+1)}a_n. \tag{1.10}$$

Just as in the case of the equation of simple harmonic motion, the first two coefficients a_0 and a_1 in the power series can be determined from the initial conditions,

$$y(0) = a_0, \qquad \frac{dy}{dx}(0) = a_1.$$

The recursion formula can be used to determine the remaining coefficients in the power series. Indeed it follows from (1.10) with $n = 0$ that

$$a_2 = -\frac{2p}{2 \cdot 1}a_0.$$

Similarly, it follows from (1.10) with $n = 1$ that

$$a_3 = \frac{2-2p}{3 \cdot 2}a_1 = -\frac{2(p-1)}{3!}a_1,$$

and with $n = 2$ that

$$a_4 = -\frac{4-2p}{4 \cdot 3}a_2 = \frac{2(2-p)}{4 \cdot 3}\frac{-2p}{2}a_0 = \frac{2^2p(p-2)}{4!}a_0.$$

Continuing in this manner, we find that

$$a_5 = \frac{6-2p}{5 \cdot 4}a_3 = \frac{2(3-p)}{5 \cdot 4}\frac{2(1-p)}{3!}a_1 = \frac{2^2(p-1)(p-3)}{5!}a_1,$$

$$a_6 = \frac{8-2p}{6\cdot 5\cdot 2}a_4 = \frac{2(3-p)}{6\cdot 5}\frac{2^2(p-2)p}{4!}a_0 = -\frac{2^3 p(p-2)(p-4)}{6!}a_0,$$

and so forth. Thus we find that

$$y = a_0\left[1 - \frac{2p}{2!}x^2 + \frac{2^2 p(p-2)}{4!}x^4 - \frac{2^3 p(p-2)(p-4)}{6!}x^6 + \cdots\right]$$
$$+a_1\left[x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5\right.$$
$$\left. - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \cdots\right].$$

We can now write the general solution to Hermite's equation in the form

$$y = a_0 y_0(x) + a_1 y_1(x),$$

where

$$y_0(x) = 1 - \frac{2p}{2!}x^2 + \frac{2^2 p(p-2)}{4!}x^4 - \frac{2^3 p(p-2)(p-4)}{6!}x^6 + \cdots$$

and

$$y_1(x) = x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \cdots .$$

For a given choice of the parameter p, we could use the power series to construct tables of values for the functions $y_0(x)$ and $y_1(x)$. Tables of values for these functions are found in many "handbooks of mathematical functions." In the language of linear algebra, we say that $y_0(x)$ and $y_1(x)$ form a basis for the space of solutions to Hermite's equation.

When p is a positive integer, one of the two power series will collapse, yielding a polynomial solution to Hermite's equation. These polynomial solutions are known as *Hermite polynomials*.

Another Example. *Legendre's differential equation* is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0, \tag{1.11}$$

where p is a parameter. This equation is very useful for treating spherically symmetric potentials in the theories of Newtonian gravitation and in electricity and magnetism.

To apply our theorem, we need to divide by $1-x^2$ to obtain

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{p(p+1)}{1-x^2}y = 0.$$

Thus we have

$$P(x) = -\frac{2x}{1-x^2}, \qquad Q(x) = \frac{p(p+1)}{1-x^2}.$$

Now from the preceding section, we know that the power series

$$1 + u + u^2 + u^3 + \cdots \qquad \text{converges to} \qquad \frac{1}{1-u}$$

for $|u| < 1$. If we substitute $u = x^2$, we can conclude that

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

the power series converging when $|x| < 1$. It follows quickly that

$$P(x) = -\frac{2x}{1-x^2} = -2x - 2x^3 - 2x^5 - \cdots$$

and

$$Q(x) = \frac{p(p+1)}{1-x^2} = p(p+1) + p(p+1)x^2 + p(p+1)x^4 + \cdots .$$

Both of these functions have power series expansions about $x_0 = 0$ which converge for $|x| < 1$. Hence our theorem implies that any solution to Legendre's equation will be expressible as a power series about $x_0 = 0$ which converges for $|x| < 1$. However, we might suspect that the solutions to Legendre's equation to exhibit some unpleasant behaviour near $x = \pm 1$. Experimentation with numerical solutions to Legendre's equation would show that these suspicions are justified—solutions to Legendre's equation will usually blow up as $x \to \pm 1$.

Indeed, it can be shown that when p is an integer, Legendre's differential equation has a nonzero polynomial solution which is well-behaved for all x, but solutions which are not constant multiples of these *Legendre polynomials* blow up as $x \to \pm 1$.

Exercises:

1.2.1. We would like to use the power series method to find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 12y = 0,$$

which is very similar to Hermite's equation. So we assume the solution is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

a power series centered at 0, and determine the coefficients a_n.

a. As a first step, find the recursion formula for a_{n+2} in terms of a_n.

b. The coefficients a_0 and a_1 will be determined by the initial conditions. Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \le n \le 9$.

c. Find a nonzero polynomial solution to this differential equation.

d. Find a basis for the space of solutions to the equation.

e. Find the solution to the initial value problem

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 12y = 0, \qquad y(0) = 0, \qquad \frac{dy}{dx}(0) = 1.$$

f. To solve the differential equation

$$\frac{d^2y}{dx^2} - 4(x-3)\frac{dy}{dx} + 12y = 0,$$

it would be most natural to assume that the solution has the form

$$y = \sum_{n=0}^{\infty} a_n (x-3)^n.$$

Use this idea to find a polynomial solution to the differential equation

$$\frac{d^2y}{dx^2} - 4(x-3)\frac{dy}{dx} + 12y = 0.$$

1.2.2. We want to use the power series method to find the general solution to Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0.$$

Once again our approach is to assume our solution is a power series centered at 0 and determine the coefficients in this power series.

a. As a first step, find the recursion formula for a_{n+2} in terms of a_n.

b. Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

c. Find a nonzero polynomial solution to this differential equation, in the case where $p = 3$.

d. Find a basis for the space of solutions to the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 12y = 0.$$

1.2.3. The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0,$$

where p is a constant, is known as *Chebyshev's equation*. It can be rewritten in the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \qquad \text{where} \qquad P(x) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{p^2}{1-x^2}.$$

a. If $P(x)$ and $Q(x)$ are represented as power series about $x_0 = 0$, what is the radius of convergence of these power series?

b. Assuming a power series centered at 0, find the recursion formula for a_{n+2} in terms of a_n.

c. Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

d. In the special case where $p = 3$, find a nonzero polynomial solution to this differential equation.

e. Find a basis for the space of solutions to

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 9y = 0.$$

1.2.4. The differential equation

$$\left(-\frac{d^2}{dx^2} + x^2\right) z = \lambda z \tag{1.12}$$

arises when treating the quantum mechanics of simple harmonic motion.

a. Show that making the substitution $z = e^{-x^2/2}y$ transforms this equation into Hermite's differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\lambda - 1)y = 0.$$

b. Show that if $\lambda = 2n+1$ where n is a nonnegative integer, (1.12) has a solution of the form $z = e^{-x^2/2}P_n(x)$, where $P_n(x)$ is a polynomial.

1.3 Singular points

Our ultimate goal is to give a mathematical description of the vibrations of a circular drum. For this, we will need to solve Bessel's equation, a second-order homogeneous linear differential equation with a "singular point" at 0.

A point x_0 is called an *ordinary point* for the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{1.13}$$

if the coefficients $P(x)$ or $Q(x)$ are both real analytic at $x = x_0$, or equivalently, both $P(x)$ or $Q(x)$ have power series expansions about $x = x_0$ with positive radius of convergence. In the opposite case, we say that x_0 is a *singular point*; thus x_0 is a singular point if at least one of the coefficients $P(x)$ or $Q(x)$ fails to be real analytic at $x = x_0$. A singular point is said to be *regular* if

$$(x - x_0)P(x) \qquad \text{and} \qquad (x - x_0)^2Q(x)$$

are real analytic.

For example, $x_0 = 1$ is a singular point for Legendre's equation

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{p(p+1)}{1-x^2}y = 0,$$

because $1 - x^2 \to 0$ as $x \to 1$ and hence the quotients

$$\frac{2x}{1-x^2} \quad \text{and} \quad \frac{p(p+1)}{1-x^2}$$

blow up as $x \to 1$, but it is a regular singular point because

$$(x-1)P(x) = (x-1)\frac{-2x}{1-x^2} = \frac{2x}{x+1}$$

and

$$(x-1)^2Q(x) = (x-1)^2\frac{p(p+1)}{1-x^2} = \frac{p(p+1)(1-x)}{1+x}$$

are both real analytic at $x_0 = 1$.

The point of these definitions is that in the case where $x = x_0$ is a regular singular point, a modification of the power series method can still be used to find solutions.

Theorem of Frobenius. *If x_0 is a regular singular point for the differential equation*

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

then this differential equation has at least one nonzero solution of the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n, \tag{1.14}$$

where r is a constant, which may be complex. If $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ have power series which converge for $|x-x_0| < R$ then the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

will also converge for $|x-x_0| < R$.

We will call a solution of the form (1.14) a *generalized power series solution*. Unfortunately, the theorem guarantees only one generalized power series solution, not a basis. In fortuitous cases, one can find a basis of generalized power series solutions, but not always. The method of finding generalized power series solutions to (1.13) in the case of regular singular points is called the *Frobenius method*.[2]

[2]For more discussion of the Frobenius method as well as many of the other techniques touched upon in this chapter we refer the reader to George F. Simmons, *Differential equations with applications and historical notes*, second edition, McGraw-Hill, New York, 1991.

The simplest differential equation to which the Theorem of Frobenius applies is the Cauchy-Euler equidimensional equation. This is the special case of (1.13) for which

$$P(x) = \frac{p}{x}, \qquad Q(x) = \frac{q}{x^2},$$

where p and q are constants. Note that

$$xP(x) = p \qquad \text{and} \qquad x^2 Q(x) = q$$

are real analytic, so $x = 0$ is a regular singular point for the Cauchy-Euler equation as long as either p or q is nonzero.

The Frobenius method is quite simple in the case of Cauchy-Euler equations. Indeed, in this case, we can simply take $y(x) = x^r$, substitute into the equation and solve for r. Often there will be two linearly independent solutions $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ of this special form. In this case, the general solution is given by the superposition principle as

$$y = c_1 x^{r_1} + c_2 x^{r_2}.$$

For example, to solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0,$$

we set $y = x^r$ and differentiate to show that

$$\begin{aligned} dy/dx &= r x^{r-1} & &\Rightarrow x(dy/dx) = r x^r, \\ d^2y/dx^2 &= r(r-1)x^{r-2} & &\Rightarrow x^2(d^2y/dx^2) = r(r-1)x^r. \end{aligned}$$

Substitution into the differential equation yields

$$r(r-1)x^r + 4r x^r + 2x^r = 0,$$

and dividing by x^r yields

$$r(r-1) + 4r + 2 = 0 \qquad \text{or} \qquad r^2 + 3r + 2 = 0.$$

The roots to this equation are $r = -1$ and $r = -2$, so the general solution to the differential equation is

$$y = c_1 x^{-1} + c_2 x^{-2} = \frac{c_1}{x} + \frac{c_2}{x^2}.$$

Note that the solutions $y_1(x) = x^{-1}$ and $y_2(x) = x^{-2}$ can be rewritten in the form

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2(x) = x^{-2} \sum_{n=0}^{\infty} b_n x^n,$$

where $a_0 = b_0 = 1$ and all the other a_n's and b_n's are zero, so both of these solutions are generalized power series solutions.

On the other hand, if this method is applied to the differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0,$$

we obtain

$$r(r-1) + 3r + 1 = r^2 + 2r + 1,$$

which has a repeated root. In this case, we obtain only a one-parameter family of solutions

$$y = cx^{-1}.$$

Fortunately, there is a trick that enables us to handle this situation, the so-called method of variation of parameters. In this context, we replace the parameter c by a variable $v(x)$ and write

$$y = v(x)x^{-1}.$$

Then

$$\frac{dy}{dx} = v'(x)x^{-1} - v(x)x^{-2}, \qquad \frac{d^2y}{dx^2} = v''(x)x^{-1} - 2v'(x)x^{-2} + 2v(x)x^{-3}.$$

Substitution into the differential equation yields

$$x^2(v''(x)x^{-1} - 2v'(x)x^{-2} + 2v(x)x^{-3}) + 3x(v'(x)x^{-1} - v(x)x^{-2}) + v(x)x^{-1} = 0,$$

which quickly simplifies to yield

$$xv''(x) + v'(x) = 0, \qquad \frac{v''}{v'} = -\frac{1}{x}, \qquad \log|v'| = -\log|x| + a, \qquad v' = \frac{c_2}{x},$$

where a and c_2 are constants of integration. A further integration yields

$$v = c_2 \log|x| + c_1, \qquad \text{so} \qquad y = (c_2 \log|x| + c_1)x^{-1},$$

and we obtain the general solution

$$y = c_1 \frac{1}{x} + c_2 \frac{\log|x|}{x}.$$

In this case, only one of the basis elements in the general solution is a generalized power series.

For equations which are not of Cauchy-Euler form the Frobenius method is more involved. Let us consider the example

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0, \tag{1.15}$$

which can be rewritten as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \qquad \text{where} \qquad P(x) = \frac{1}{2x}, \qquad Q(x) = \frac{1}{2x}.$$

One easily checks that $x = 0$ is a regular singular point. We begin the Frobenius method by assuming that the solution has the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

and

$$2x\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1}.$$

Substitution into the differential equation yields

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

which simplifies to

$$x^r \left[\sum_{n=0}^{\infty} (2n+2r-1)(n+r) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \right] = 0.$$

We can divide by x^r, and separate out the first term from the first summation, obtaining

$$(2r-1)ra_0 x^{-1} + \sum_{n=1}^{\infty} (2n+2r-1)(n+r) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

If we let $n = m + 1$ in the first infinite sum, this becomes

$$(2r-1)ra_0 x^{-1} + \sum_{m=0}^{\infty} (2m+2r+1)(m+r+1) a_{m+1} x^m + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Finally, we replace m by n, obtaining

$$(2r-1)ra_0 x^{-1} + \sum_{n=0}^{\infty} (2n+2r+1)(n+r+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

The coefficient of each power of x must be zero. In particular, we must have

$$(2r-1)ra_0 = 0, \qquad (2n+2r+1)(n+r+1)a_{n+1} + a_n = 0. \tag{1.16}$$

If $a_0 = 0$, then all the coefficients must be zero from the second of these equations, and we don't get a nonzero solution. So we must have $a_0 \neq 0$ and hence

$$(2r-1)r = 0.$$

This is called the *indicial equation*. In this case, it has two roots

$$r_1 = 0, \qquad r_2 = \frac{1}{2}.$$

The second half of (1.16) yields the recursion formula

$$a_{n+1} = -\frac{1}{(2n+2r+1)(n+r+1)} a_n, \qquad \text{for} \qquad n \geq 0.$$

We can try to find a generalized power series solution for either root of the indicial equation. If $r = 0$, the recursion formula becomes

$$a_{n+1} = -\frac{1}{(2n+1)(n+1)} a_n.$$

Given $a_0 = 1$, we find that

$$a_1 = -1, \qquad a_2 = -\frac{1}{3 \cdot 2} a_1 = \frac{1}{3 \cdot 2},$$

$$a_3 = -\frac{1}{5 \cdot 3} a_2 = -\frac{1}{(5 \cdot 3)(3 \cdot 2)}, \qquad a_4 = -\frac{1}{7 \cdot 4} a_3 = \frac{1}{(7 \cdot 5 \cdot 3)4!},$$

and so forth. In general, we would have

$$a_n = (-1)^n \frac{1}{(2n-1)(2n-3) \cdots 1 \cdot n!}.$$

One of the generalized power series solution to (1.15) is

$$\begin{aligned} y_1(x) &= x^0 \left[1 - x + \frac{1}{3 \cdot 2} x^2 - \frac{1}{(5 \cdot 3)(3!)} x^3 + \frac{1}{(7 \cdot 5 \cdot 3)4!} x^4 - \cdots \right] \\ &= 1 - x + \frac{1}{3 \cdot 2} x^2 - \frac{1}{(5 \cdot 3)(3!)} x^3 + \frac{1}{(7 \cdot 5 \cdot 3)4!} x^4 - \cdots . \end{aligned}$$

If $r = 1/2$, the recursion formula becomes

$$a_{n+1} = -\frac{1}{(2n+2)(n+(1/2)+1)} a_n = -\frac{1}{(n+1)(2n+3)} a_n.$$

Given $a_0 = 1$, we find that

$$a_1 = -\frac{1}{3}, \qquad a_2 = -\frac{1}{2 \cdot 5} a_1 = \frac{1}{2 \cdot 5 \cdot 3},$$

$$a_3 = -\frac{1}{3 \cdot 7} a_2 = -\frac{1}{3! \cdot (7 \cdot 5 \cdot 3)},$$

and in general,

$$a_n = (-1)^n \frac{1}{n!(2n+1)(2n-1) \cdots 1 \cdot n!}.$$

We thus obtain a second generalized power series solution to (1.15):

$$y_2(x) = x^{1/2}\left[1 - \frac{1}{3}x + \frac{1}{2\cdot 5\cdot 3}x^2 - \frac{1}{3!\cdot(7\cdot 5\cdot 3)}x^3 + \cdots\right].$$

The general solution to (1.15) is a superposition of $y_1(x)$ and $y_2(x)$:

$$y = c_1\left[1 - x + \frac{1}{3\cdot 2}x^2 - \frac{1}{(5\cdot 3)(3!)}x^3 + \frac{1}{(7\cdot 5\cdot 3)4!}x^4 - \cdots\right]$$
$$+c_2\sqrt{x}\left[1 - \frac{1}{3}x + \frac{1}{2\cdot 5\cdot 3}x^2 - \frac{1}{3!\cdot(7\cdot 5\cdot 3)}x^3 + \cdots\right].$$

We obtained two linearly independent generalized power series solutions in this case, but this does not always happen. If the roots of the indicial equation differ by an integer, we may obtain only one generalized power series solution. In that case, a second independent solution can then be found by variation of parameters, just as we saw in the case of the Cauchy-Euler equidimensional equation.

Exercises:

1.3.1. For each of the following differential equations, determine whether $x = 0$ is ordinary or singular. If it is singular, determine whether it is regular or not.

a. $y'' + xy' + (1 - x^2)y = 0$.

b. $y'' + (1/x)y' + (1 - (1/x^2))y = 0$.

c. $x^2y'' + 2xy' + (\cos x)y = 0$.

d. $x^3y'' + 2xy' + (\cos x)y = 0$.

1.3.2. Find the general solution to each of the following Cauchy-Euler equations:

a. $x^2d^2y/dx^2 - 2xdy/dx + 2y = 0$.

b. $x^2d^2y/dx^2 - xdy/dx + y = 0$.

c. $x^2d^2y/dx^2 - xdy/dx + 10y = 0$.

(Hint: Use the formula

$$x^{a+bi} = x^ax^{bi} = x^a(e^{\log x})^{bi} = x^ae^{ib\log x} = x^a[\cos(b\log x) + i\sin(b\log x)]$$

to simplify the answer.)

1.3.3. We want to find generalized power series solutions to the differential equation

$$3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

by the method of Frobenius. Our procedure is to find solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r and the a_n's are constants.

a. Determine the indicial equation and the recursion formula.

b. Find two linearly independent generalized power series solutions.

1.3.4. To find generalized power series solutions to the differential equation

$$2x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + xy = 0$$

by the method of Frobenius, we assume the solution has the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r and the a_n's are constants.

a. Determine the indicial equation and the recursion formula.

b. Find two linearly independent generalized power series solutions.

1.4 Bessel's differential equation

Our next goal is to apply the Frobenius method to *Bessel's equation*,

$$x \frac{d}{dx}\left(x \frac{dy}{dx} \right) + (x^2 - p^2) y = 0, \tag{1.17}$$

an equation which is needed to analyze the vibrations of a circular drum, as we mentioned before. Here p is a parameter, which will be a nonnegative integer in the vibrating drum problem. Using the Leibniz rule for differentiating a product, we can rewrite Bessel's equation in the form

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2) y = 0$$

or equivalently as

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) = 0,$$

where

$$P(x) = \frac{1}{x} \qquad \text{and} \qquad Q(x) = \frac{x^2 - p^2}{x^2}.$$

Since

$$xP(x) = 1 \qquad \text{and} \qquad x^2 Q(x) = x^2 - p^2,$$

we see that $x = 0$ is a regular singular point, so the Frobenius theorem implies that there exists a nonzero generalized power series solution to (1.17).

To find such a solution, we start as in the previous section by assuming that

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad x\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r},$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r-1},$$

and thus

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r}. \tag{1.18}$$

On the other hand,

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+r},$$

where we have set $m = n + 2$. Replacing m by n then yields

$$x^2 y = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}. \tag{1.19}$$

Finally, we have,

$$-p^2 y = -\sum_{n=0}^{\infty} p^2 a_n x^{n+r}. \tag{1.20}$$

Adding up (1.18), (1.19), and (1.20), we find that if y is a solution to (1.17),

$$\sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \sum_{n=0}^{\infty} p^2 a_n x^{n+r} = 0.$$

This simplifies to yield

$$\sum_{n=0}^{\infty} [(n+r)^2 - p^2] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0,$$

or after division by x^r,

$$\sum_{n=0}^{\infty} [(n+r)^2 - p^2] a_n x^{n} + \sum_{n=2}^{\infty} a_{n-2} x^{n} = 0.$$

Thus we find that

$$(r^2 - p^2)a_0 + [(r+1)^2 - p^2]a_1 x + \sum_{n=2}^{\infty} \{[(n+r)^2 - p^2]a_n + a_{n-2}\}x^n = 0.$$

The coefficient of each power of x must be zero, so

$$(r^2 - p^2)a_0 = 0, \quad [(r+1)^2 - p^2]a_1 = 0, \quad [(n+r)^2 - p^2]a_n + a_{n-2} = 0 \quad \text{for } n \geq 2.$$

Since we want a_0 to be nonzero, r must satisfy the *indicial equation*

$$(r^2 - p^2) = 0,$$

which implies that $r = \pm p$. Let us assume without loss of generality that $p \geq 0$ and take $r = p$. Then

$$[(p+1)^2 - p^2]a_1 = 0 \qquad \Rightarrow \qquad (2p+1)a_1 = 0 \qquad \Rightarrow \qquad a_1 = 0.$$

Finally,

$$[(n+p)^2 - p^2]a_n + a_{n-2} = 0 \qquad \Rightarrow \qquad [n^2 + 2np]a_n + a_{n-2} = 0,$$

which yields the recursion formula

$$a_n = -\frac{1}{2np + n^2} a_{n-2}. \tag{1.21}$$

The recursion formula implies that $a_n = 0$ if n is odd.

In the special case where p is a nonnegative integer, we will get a genuine power series solution to Bessel's equation (1.17). Let us focus now on this important case. If we set

$$a_0 = \frac{1}{2^p p!},$$

we obtain

$$a_2 = \frac{-a_0}{4p+4} = -\frac{1}{4(p+1)} \frac{1}{2^p p!} = (-1)\left(\frac{1}{2}\right)^{p+2} \frac{1}{1!(p+1)!},$$

$$a_4 = \frac{-a_2}{8p+16} = \frac{1}{8(p+2)} \left(\frac{1}{2}\right)^{p+2} \frac{1}{1!(p+1)!} =$$

$$= \frac{1}{2(p+2)} \left(\frac{1}{2}\right)^{p+4} \frac{1}{1!(p+1)!} = (-1)^2 \left(\frac{1}{2}\right)^{p+4} \frac{1}{2!(p+2)!},$$

and so forth. The general term is

$$a_{2m} = (-1)^m \left(\frac{1}{2}\right)^{p+2m} \frac{1}{m!(p+m)!}.$$

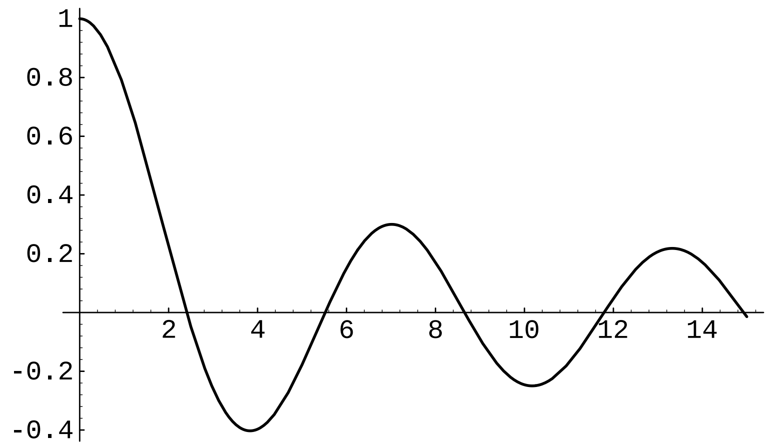

Figure 1.1: Graph of the Bessel function $J_0(x)$.

Thus we finally obtain the power series solution

$$y = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(p+m)!} \left(\frac{x}{2}\right)^{2m}.$$

The function defined by the power series on the right-hand side is called the p-th *Bessel function of the first kind*, and is denoted by the symbol $J_p(x)$. For example,

$$J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{x}{2}\right)^{2m}.$$

Using the comparison and ratio tests, we can show that the power series expansion for $J_p(x)$ has infinite radius of convergence. Thus when p is an integer, Bessel's equation has a nonzero solution which is real analytic at $x = 0$.

Bessel functions are so important that Mathematica includes them in its library of built-in functions.[3] Mathematica represents the Bessel functions of the first kind symbolically by `BesselJ[n,x]`. Thus to plot the Bessel function $J_n(x)$ on the interval $[0, 15]$ one simply types in

```
n=0; Plot[ BesselJ[n,x], {x,0,15}]
```

and a plot similar to that of Figure 1.1 will be produced. Similarly, we can plot $J_n(x)$, for $n = 1, 2, 3 \ldots$. Note that the graph of $J_0(x)$ suggests that it has infinitely many positive zeros.

On the open interval $0 < x < \infty$, Bessel's equation has a two-dimensional space of solutions. However, it turns out that when p is a nonnegative integer, a second solution, linearly independent from the Bessel function of the first kind,

[3]For a very brief introduction to Mathematica, the reader can refer to Appendix A.

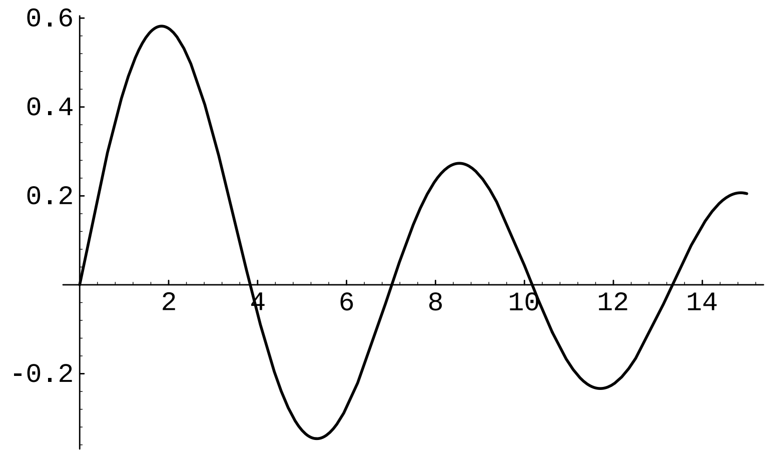

Figure 1.2: Graph of the Bessel function $J_1(x)$.

cannot be obtained directly by the generalized power series method that we have presented. To obtain a basis for the space of solutions, we can, however, apply the method of variation of parameters just as we did in the previous section for the Cauchy-Euler equation; namely, we can set

$$y = v(x)J_p(x),$$

substitute into Bessel's equation and solve for $v(x)$. If we were to carry this out in detail, we would obtain a second solution linearly independent from $J_p(x)$. Appropriately normalized, his solution is often denoted by $Y_p(x)$ and called the p-th *Bessel function of the second kind*. Unlike the Bessel function of the first kind, this solution is not well-behaved near $x = 0$.

To see why, suppose that $y_1(x)$ and $y_2(x)$ is a basis for the solutions on the interval $0 < x < \infty$, and let $W(y_1, y_2)$ be their *Wronskian*, defined by

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_1'(x) \\ y_2(x) & y_2'(x) \end{vmatrix}.$$

This Wronskian must satisfy the first order equation

$$\frac{d}{dx}(xW(y_1, y_2)(x)) = 0,$$

as one verifies by a direct calculation:

$$x\frac{d}{dx}(xy_1y_2' - xy_2y_1') = y_1x\frac{d}{dx}(xy_2') - y_2x\frac{d}{dx}(xy_1')$$
$$= -(x^2 - n^2)(y_1y_2 - y_2y_1) = 0.$$

Thus

$$xW(y_1, y_2)(x) = c, \qquad \text{or} \qquad W(y_1, y_2)(x) = \frac{c}{x},$$

where c is a nonzero constant, an expression which is unbounded as $x \to 0$. It follows that two linearly independent solutions $y_1(x)$ and $y_2(x)$ to Bessel's equation cannot both be well-behaved at $x = 0$.

Let us summarize what we know about the space of solutions to Bessel's equation in the case where p is an integer:

- There is a one-dimensional space of real analytic solutions to (1.17), which are well-behaved as $x \to 0$.
- This one-dimensional space is generated by a function $J_p(x)$ which is given by the explicit power series formula

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(p+m)!} \left(\frac{x}{2}\right)^{2m}.$$

Exercises:

1.4.1. Using the explicit power series formulae for $J_0(x)$ and $J_1(x)$ show that

$$\frac{d}{dx} J_0(x) = -J_1(x) \qquad \text{and} \qquad \frac{d}{dx}(xJ_1(x)) = xJ_0(x).$$

1.4.2. The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + p^2)y = 0$$

is sometimes called a *modified Bessel equation*. Find a generalized power series solution to this equation in the case where p is an integer. (Hint: The power series you obtain should be very similar to the power series for $J_p(x)$.)

1.4.3. Show that the functions

$$y_1(x) = \frac{1}{\sqrt{x}} \cos x \qquad \text{and} \qquad y_2(x) = \frac{1}{\sqrt{x}} \sin x$$

are solutions to Bessel's equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - p^2)y = 0,$$

in the case where $p = 1/2$. Hence the general solution to Bessel's equation in this case is

$$y = c_1 \frac{1}{\sqrt{x}} \cos x + c_2 \frac{1}{\sqrt{x}} \sin x.$$

1.4.4. To obtain a nice expression for the generalized power series solution to Bessel's equation in the case where p is not an integer, it is convenient to use the *gamma function* defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

a. Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$.

b. Show that $\Gamma(1) = 1$.

c. Show that

$$\Gamma(n+1) = n! = n(n-1)\cdots 2 \cdot 1,$$

when n is a positive integer.

d. Set

$$a_0 = \frac{1}{2^p \Gamma(p+1)},$$

and use the recursion formula (1.21) to obtain the following generalized power series solution to Bessel's equation (1.17) for general choice of p:

$$y = J_p(x) = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m}.$$

Chapter 2

Symmetry and Orthogonality

2.1 Eigenvalues of symmetric matrices

Before proceeding further, we need to review and extend some notions from vectors and matrices (linear algebra), which the student should have studied in an earlier course. In particular, we will need the amazing fact that the eigenvalue-eigenvector problem for an $n \times n$ matrix A simplifies considerably when the matrix is symmetric.

An $n \times n$ matrix A is said to be *symmetric* if it is equal to its transpose A^T. Examples of symmetric matrices include

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 3-\lambda & 6 & 5 \\ 6 & 1-\lambda & 0 \\ 5 & 0 & 8-\lambda \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

Alternatively, we could say that an $n \times n$ matrix A is symmetric if and only if

$$\mathbf{x} \cdot (A\mathbf{y}) = (A\mathbf{x}) \cdot \mathbf{y}. \tag{2.1}$$

for every choice of n-vectors $\mathbf{x}$ and $\mathbf{y}$. Indeed, since $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$, equation (2.1) can be rewritten in the form

$$\mathbf{x}^T A\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y},$$

which holds for all $\mathbf{x}$ and $\mathbf{y}$ if and only if $A = A^T$.

On the other hand, an $n \times n$ real matrix B is *orthogonal* if its transpose is equal to its inverse, $B^T = B^{-1}$. Alternatively, an $n \times n$ matrix

$$B = (\mathbf{b}_1\mathbf{b}_2 \cdots \mathbf{b}_n)$$

is orthogonal if its column vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ satisfy the relations

$$\begin{array}{llll} \mathbf{b}_1 \cdot \mathbf{b}_1 = 1, & \mathbf{b}_1 \cdot \mathbf{b}_2 = 0, & \cdots, & \mathbf{b}_1 \cdot \mathbf{b}_n = 0, \\ & \mathbf{b}_2 \cdot \mathbf{b}_2 = 1, & \cdots, & \mathbf{b}_2 \cdot \mathbf{b}_n = 0, \\ & & & \cdot \\ & & & \mathbf{b}_n \cdot \mathbf{b}_n = 1. \end{array}$$

Using this latter criterion, we can easily check that, for example, the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

are orthogonal. Note that since

$$B^T B = I \quad \Rightarrow \quad (\det B)^2 = (\det B^T)(\det B) = \det(B^T B) = 1,$$

the determinant of an orthogonal matrix is always ± 1.

Recall that the *eigenvalues* of an $n \times n$ matrix A are the roots of the polynomial equation

$$\det(A - \lambda I) = 0.$$

For each eigenvalue λ_i, the corresponding *eigenvectors* are the nonzero solutions $\mathbf{x}$ to the linear system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

For a general $n \times n$ matrix with real entries, the problem of finding eigenvalues and eigenvectors can be complicated, because eigenvalues need not be real (but can occur in complex conjugate pairs) and in the "repeated root" case, there may not be enough eigenvectors to construct a basis for R^n. We will see that these complications do not occur for symmetric matrices.

Spectral Theorem.[1] *Suppose that A is a symmetric $n \times n$ matrix with real entries. Then its eigenvalues are real and eigenvectors corresponding to distinct eigenvectors are orthogonal. Moreover, there is an $n \times n$ orthogonal matrix B of determinant one such that $B^{-1}AB = B^T AB$ is diagonal.*

Sketch of proof: The reader may want to skip our sketch of the proof at first, returning after studying some of the examples presented later in this section. We will assume the following two facts, which are proven rigorously in advanced courses on mathematical analysis:

1. Any continuous function on a sphere (of arbitrary dimension) assumes its maximum and minimum values.

2. The points at which the maximum and minimum values are assumed can be found by the method of Lagrange multipliers (a method usually discussed in vector calculus courses).

[1] This is called the "spectral theorem" because the spectrum is another name for the set of eigenvalues of a matrix.

The equation of the sphere S^{n-1} in R^n is

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1, \quad \text{or} \quad \mathbf{x}^T\mathbf{x} = 1, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}.$$

We let

$$g(\mathbf{x}) = g(x_1, x_2, \ldots, x_n) = \mathbf{x}^T\mathbf{x} - 1,$$

so that the equation of the sphere is given by the constraint equation $g(\mathbf{x}) = 0$. Our approach consists of finding of finding the point on S^{n-1} at which the function

$$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) = \mathbf{x}^T A\mathbf{x}$$

assumes its maximum values.

To find this maximum using Lagrange multipliers, we look for "critical points" for the function

$$H(\mathbf{x}, \lambda) = H(x_1, x_2, \ldots, x_n, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}).$$

These are points at which

$$\nabla f(x_1, x_2, \ldots, x_n) = \lambda \nabla g(x_1, x_2, \ldots, x_n), \quad \text{and} \quad g(x_1, x_2, \ldots, x_n) = 0.$$

In other words, these are the points on the sphere at which the gradient of f is a multiple of the gradient of g, or the points on the sphere at which the gradient of f is perpendicular to the sphere.

If we set

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{pmatrix},$$

a short calculation shows that at the point where f assumes its maximum,

$$\frac{\partial H}{\partial x_i} = 2a_{i1}x_1 + 2a_{i2}x_2 + \cdots + 2a_{in}x_n - 2\lambda x_i = 0,$$

or equivalently,

$$A\mathbf{x} - \lambda\mathbf{x} = 0.$$

We also obtain the condition

$$\frac{\partial H}{\partial \lambda} = -g(\mathbf{x}) = 0,$$

which is just our constraint. Thus the point on the sphere at which f assumes its maximum is a unit-length eigenvector $\mathbf{b}_1$, the eigenvalue being the value λ_1 of the variable λ.

Let W be the "linear subspace" of R^n defined by the homogeneous linear equation $\mathbf{b}_1 \cdot \mathbf{x} = 0$. The intersection $S^{n-1} \cap W$ is a sphere of dimension $n - 2$.

We next use the method of Lagrange multipliers to find a point on $S^{n-1} \cap W$ at which f assumes its maximum. To do this, we let

$$g_1(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - 1, \qquad g_2(\mathbf{x}) = \mathbf{b}_1 \cdot \mathbf{x}.$$

The maximum value of f on $S^{n-1} \cap W$ will be assumed at a critical point for the function

$$H(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) - \lambda g_1(\mathbf{x}) - \mu g_2(\mathbf{x}).$$

This time, differentiation shows that

$$\frac{\partial H}{\partial x_i} = 2a_{i1}x_1 + 2a_{i2}x_2 + \cdots + 2a_{in}x_n - 2\lambda x_i - \mu b_i = 0,$$

or equivalently,

$$A\mathbf{x} - \lambda\mathbf{x} - \mu\mathbf{b}_1 = 0. \tag{2.2}$$

It follows from the constraint equation $\mathbf{b}_1 \cdot \mathbf{x} = 0$ that

$$\begin{aligned} \mathbf{b}_1 \cdot (A\mathbf{x}) = \mathbf{b}_1^T(A\mathbf{x}) = (\mathbf{b}_1^T A)\mathbf{x} = (\mathbf{b}_1^T A^T)\mathbf{x} \\ = (A\mathbf{b}_1)^T\mathbf{x} = (\lambda_1\mathbf{b}_1)^T\mathbf{x} = \lambda_1\mathbf{b}_1 \cdot \mathbf{x} = 0. \end{aligned}$$

Hence it follows from (2.2) that $A\mathbf{x} - \lambda\mathbf{x} = 0$. Thus if $\mathbf{b}_2$ is a point on $S^{n-1} \cap W$ at which f assumes its maximum, $\mathbf{b}_2$ must be a unit-length eigenvector for A which is perpendicular to $\mathbf{b}_1$.

Continuing in this way we finally obtain n mutually orthogonal unit-length eigenvectors $\mathbf{b}_1$, $\mathbf{b}_2$, ... , $\mathbf{b}_n$. These eigenvectors satisfy the equations

$$A\mathbf{b}_1 = \lambda_1\mathbf{b}_1, \quad A\mathbf{b}_2 = \lambda_2\mathbf{b}_2, \quad \ldots \quad A\mathbf{b}_n = \lambda_2\mathbf{b}_n,$$

which can be put together in a single matrix equation

$$\begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdot & A\mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{b}_1 & \lambda_2\mathbf{b}_2 & \cdot & \lambda_n\mathbf{b}_n \end{pmatrix},$$

or equivalently,

$$A\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdot & \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdot & \mathbf{b}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdot & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{pmatrix}.$$

If we set

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdot & \mathbf{b}_n \end{pmatrix},$$

this last equation becomes

$$AB = B\begin{pmatrix} \lambda_1 & 0 & \cdot & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{pmatrix}.$$

Of course, B is an orthogonal matrix, so it is invertible and we can solve for A, obtaining

$$A = B \begin{pmatrix} \lambda_1 & 0 & \cdot & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{pmatrix} B^{-1}, \quad \text{or} \quad B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & \cdot & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{pmatrix}.$$

We can arrange that the determinant of B be one by changing the sign of one of the columns if necessary.

A more complete proof of the theorem is presented in more advanced courses in linear algebra.[2] In any case, the method for finding the orthogonal matrix B such that $B^T AB$ is diagonal is relatively simple, at least when the eigenvalues are distinct. Simply let B be the matrix whose columns are unit-length eigenvectors for A. In the case of repeated roots, we must be careful to choose a basis of unit-length eigenvectors for each eigenspace which are perpendicular to each other.

Example. The matrix

$$A = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is symmetric, so its eigenvalues must be real. Its characteristic equation is

$$0 = \begin{vmatrix} 5-\lambda & 4 & 0 \\ 4 & 5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = [(\lambda-5)^2 - 16](\lambda - 1)$$

$$= (\lambda^2 - 10\lambda + 9)(\lambda - 1) = (\lambda - 1)^2(\lambda - 9),$$

which has the roots $\lambda_1 = 1$ with multiplicity two and $\lambda_2 = 9$ with multiplicity one.

Thus we are in the notorious "repeated root" case, which might be expected to cause problems if A were not symmetric. However, since A is symmetric, the Spectral Theorem guarantees that we can find a basis for R^3 consisting of eigenvectors for A even when the roots are repeated.

We first consider the eigenspace W_1 corresponding to the eigenvalue $\lambda_1 = 1$, which consists of the solutions to the linear system

$$\begin{pmatrix} 5-1 & 4 & 0 \\ 4 & 5-1 & 0 \\ 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0,$$

or

$$\begin{aligned} 4b_1 \quad +4b_2 \quad &= 0, \\ 4b_1 \quad +4b_2 \quad &= 0, \\ 0 \quad &= 0. \end{aligned}$$

[2] There are many excellent linear algebra texts that prove this theorem in detail; one good reference is Bill Jacob, *Linear algebra*, W. H. Freeman, New York, 1990; see Chapter 5.

The coefficient matrix of this linear system is

$$\begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the elementary row operations to this matrix yields

$$\begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the linear system is equivalent to

$$\begin{aligned} b_1 + b_2 &= 0, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

Thus W_1 is a plane with equation $b_1 + b_2 = 0$.

We need to extract two unit length eigenvectors from W_1 which are perpendicular to each other. Note that since the equation for W_1 is $b_1 + b_2 = 0$, the unit length vector

$$\mathbf{n} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

is perpendicular to W_1. Since

$$\mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_1, \quad \text{we find that} \quad \mathbf{b}_2 = \mathbf{n} \times \mathbf{b}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \in W_1.$$

The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are unit length elements of W_1 which are perpendicular to each other.

Next, we consider the eigenspace W_9 corresponding to the eigenvalue $\lambda_2 = 9$, which consists of the solutions to the linear system

$$\begin{pmatrix} 5-9 & 4 & 0 \\ 4 & 5-9 & 0 \\ 0 & 0 & 1-9 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0,$$

or

$$\begin{aligned} -4b_1 \quad +4b_2 \qquad\qquad &= 0, \\ 4b_1 \quad -4b_2 \qquad\qquad &= 0, \\ -8b_3 &= 0. \end{aligned}$$

The coefficient matrix of this linear system is

$$\begin{pmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -8 \end{pmatrix}.$$

Applying the elementary row operations to this matrix yields

$$\begin{pmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the linear system is equivalent to

$$\begin{aligned} b_1 - b_2 &= 0, \\ b_3 &= 0, \\ 0 &= 0, \end{aligned}$$

and we see that

$$W_9 = \operatorname{span} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We set

$$\mathbf{b}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Theory guarantees that the matrix

$$B = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix},$$

whose columns are the eigenvectors $\mathbf{b}_1$, $\mathbf{b}_1$, and $\mathbf{b}_3$, will satisfy

$$B^{-1}AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Moreover, since the eigenvectors we have chosen are of unit length and perpendicular to each other, the matrix B will be orthogonal.

Exercises:

2.1.1. Find a 2×2-matrix B such that B is orthogonal and $B^{-1}AB$ is diagonal, where

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}.$$

2.1.2. Find a 2×2-matrix B such that B is orthogonal and $B^{-1}AB$ is diagonal, where

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}.$$

2.1.3. Find a 3×3-matrix B such that B is orthogonal and $B^{-1}AB$ is diagonal, where

$$A = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.1.4. Find a 3×3-matrix B such that B is orthogonal and $B^{-1}AB$ is diagonal, where

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

2.1.5. Show that the $n \times n$ identity matrix I is orthogonal. Show that if B_1 and B_2 are $n \times n$ orthogonal matrices, so is the product matrix B_1B_2, as well as the inverse B_1^{-1}.

2.2 Conic sections and quadric surfaces

The theory presented in the previous section can be used to "rotate coordinates" so that conic sections or quadric surfaces can be put into "canonical form."

A *conic section* is a curve in R^2 which is described by a quadratic equation, such as the equation

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = 1, \tag{2.3}$$

where a, b and c are constants, which can be written in matrix form as

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1.$$

If we let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

we can rewrite (2.3) as

$$\mathbf{x}^T A\mathbf{x} = 1, \tag{2.4}$$

where A is a symmetric matrix.

According to the Spectral Theorem from Section 2.1, there exists a 2×2 orthogonal matrix B of determinant one such that

$$B^{-1}AB = B^TAB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

If we make a linear change of variables,

$$\mathbf{x} = B\mathbf{y},$$

then since $\mathbf{x}^T = \mathbf{y}^T B^T$, equation (2.3) is transformed into

$$\mathbf{y}^T(B^TAB)\mathbf{y} = 1, \qquad \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1,$$

or equivalently,

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1. \tag{2.5}$$

In the new coordinate system (y_1, y_2), it is easy to recognize the conic section:

- If λ_1 and λ_2 are both positive, the conic is an *ellipse*.
- If λ_1 and λ_2 have opposite signs, the conic is an *hyperbola*.
- If λ_1 and λ_2 are both negative, the conic degenerates to the *the empty set*, because the equation has no real solutions.

In the case where λ_1 and λ_2 are both positive, we can rewrite (2.5) as

$$\frac{y_1^2}{(\sqrt{1/\lambda_1})^2} + \frac{y_2^2}{(\sqrt{1/\lambda_2})^2} = 1,$$

from which we recognize that the semi-major and semi-minor axes of the ellipse are $\sqrt{1/\lambda_1}$ and $\sqrt{1/\lambda_2}$.

The matrix B which relates $\mathbf{x}$ and $\mathbf{y}$ represents a rotation of the plane. To see this, note that the first column $\mathbf{b}_1$ of B is a unit-length vector, and can therefore be written in the form

$$\mathbf{b}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

for some choice of θ. The second column $\mathbf{b}_2$ is a unit-vector perpendicular to $\mathbf{b}_1$ and hence

$$\mathbf{b}_2 = \pm \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

We must take the plus sign in this last expression, because $\det B = 1$. Thus

$$B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

or equivalently, B takes the standard basis vectors for R^2 to vectors which have been rotated counterclockwise through an angle θ. By linearity,

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

must rotate every element of R^2 counterclockwise through an angle θ. Thus once we have sketched the conic in (y_1, y_2)-coordinates, we can obtain the sketch in (x_1, x_2)-coordinates by simply rotating counterclockwise through the angle θ.

Example. Let's consider the conic

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = 1, \tag{2.6}$$

or equivalently,

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1.$$

The characteristic equation of the matrix

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$$

is

$$(5-\lambda)^2 - 9 = 0, \quad \lambda^2 - 10\lambda + 16 = 0, \quad (\lambda - 2)(\lambda - 8) = 0,$$

and the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 8$. Unit-length eigenvectors corresponding to these eigenvalues are

$$\mathbf{b}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

The proof of the theorem of Section 2.1 shows that these vectors are orthogonal to each other, and hence the matrix

$$B = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

is an orthogonal matrix such that

$$B^T A B = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}.$$

Note that B represents a counterclockwise rotation through 45 degrees. If we define new coordinates (y_1, y_2) by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

equation (2.6) will simplify to

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1,$$

or

$$2y_1^2 + 8y_2^2 = \frac{y_1^2}{(1/\sqrt{2})^2} + \frac{y_2^2}{(1/\sqrt{8})^2} = 1.$$

We recognize that this is the equation of an ellipse. The lengths of the semimajor and semiminor axes are $1/\sqrt{2}$ and $1/(2\sqrt{2})$.

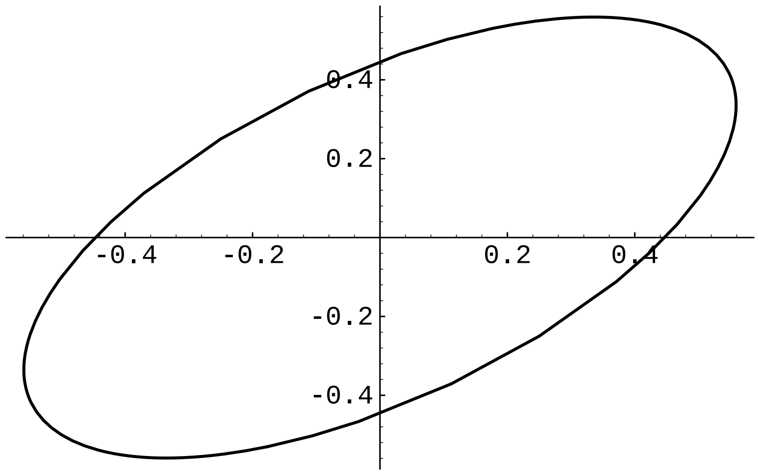

Figure 2.1: Sketch of the conic section $5x_1^2 - 6x_1x_2 + 5x_2^2 - 1 = 0$.

The same techniques can be used to sketch *quadric surfaces* in R^3, surfaces defined by an equation of the form

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1,$$

where the a_{ij}'s are constants. If we let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

we can write this in matrix form

$$\mathbf{x}^T A \mathbf{x} = 1. \tag{2.7}$$

According to the Spectral Theorem, there is a 3×3 orthogonal matrix B of determinant one such that $B^T A B$ is diagonal. We introduce new coordinates

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \text{by setting} \qquad \mathbf{x} = B\mathbf{y},$$

and equation (2.7) becomes

$$\mathbf{y}^T (B^T A B) \mathbf{y} = 1.$$

Thus after a suitable linear change of coordinates, the equation (2.7) can be put in the form

$$\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 1,$$

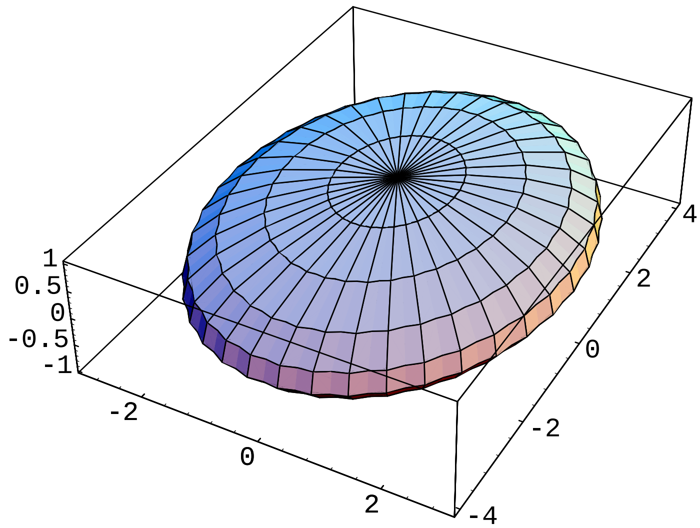

Figure 2.2: An ellipsoid.

or

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1,$$

where λ_1, λ_2, and λ_3 are the eigenvalues of A. It is relatively easy to sketch the quadric surface in the coordinates (y_1, y_2, y_3).

If the eigenvalues are all nonzero, we find that there are four cases:

- If λ_1, λ_2, and λ_3 are all positive, the quadric surface is an *ellipsoid*.
- If two of the λ's are positive and one is negative, the quadric surface is an *hyperboloid of one sheet*.
- If two of the λ's are negative and one is positive, the quadric surface is an *hyperboloid of two sheets*.
- If λ_1, λ_2, and λ_3 are all negative, the equation represents the *empty set*.

Just as in the case of conic sections, the orthogonal matrix B of determinant one which relates $\mathbf{x}$ and $\mathbf{y}$ represents a rotation. To see this, note first that since B is orthogonal,

$$(B\mathbf{x}) \cdot (B\mathbf{y}) = \mathbf{x}^T B^T B\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x} \cdot \mathbf{y}. \tag{2.8}$$

In other words, multiplication by B preserves dot products. It follows from this that the only real eigenvalues of B can be ± 1. Indeed, if $\mathbf{x}$ is an eigenvector for B corresponding to the eigenvalue λ, then

$$\lambda^2(\mathbf{x} \cdot \mathbf{x}) = (\lambda\mathbf{x}) \cdot (\lambda\mathbf{x}) = (B\mathbf{x}) \cdot (B\mathbf{x}) = \mathbf{x} \cdot \mathbf{x},$$

so division by $\mathbf{x} \cdot \mathbf{x}$ yields $\lambda^2 = 1$.

Since $\det B = 1$ and $\det B$ is the product of the eigenvalues, if all of the eigenvalues are real, $\lambda = 1$ must occur as an eigenvalue. On the other hand, non-real eigenvalues must occur in complex conjugate pairs, so if there is a non-real eigenvalue $\mu + i\nu$, then there must be another non-real eigenvalue $\mu - i\nu$ together with a real eigenvalue λ. In this case,

$$1 = \det B = \lambda(\mu + i\nu)(\mu - i\nu) = \lambda(\mu^2 + \nu^2).$$

Since $\lambda = \pm 1$, we conclude that $\lambda = 1$ must occur as an eigenvalue also in this case.

In either case, $\lambda = 1$ is an eigenvalue and

$$W_1 = \{\mathbf{x} \in \mathsf{R}^3 : B\mathbf{x} = \mathbf{x}\}$$

is nonzero. It is easily verified that if $\dim W_1$ is larger than one, then B must be the identity. Thus if $B \neq I$, there is a one-dimensional subspace W_1 of R^3 which is left fixed under multiplication by B. This is the *axis* of rotation.

Let $W_1^\perp$ denote the orthogonal complement to W_1. If $\mathbf{x}$ is a nonzero element of W_1 and $\mathbf{y} \in W_1^\perp$, it follows from (2.8) that

$$(B\mathbf{y}) \cdot \mathbf{x} = (B\mathbf{y}) \cdot (B\mathbf{x}) = \mathbf{y}^T B^T B\mathbf{x} = \mathbf{y}^T I\mathbf{x} = \mathbf{y} \cdot \mathbf{x} = 0,$$

so $B\mathbf{y} \in W_1^\perp$. Let $\mathbf{y}$, $\mathbf{z}$ be elements of $W_1^\perp$ such that

$$\mathbf{y} \cdot \mathbf{y} = 1, \quad \mathbf{y} \cdot \mathbf{z} = 0, \quad \mathbf{z} \cdot \mathbf{z} = 1;$$

we could say that $\{\mathbf{y}, \mathbf{z}\}$ form an orthonormal basis for $W_1^\perp$. By (2.8),

$$(B\mathbf{y}) \cdot (B\mathbf{y}) = 1, \quad (B\mathbf{y}) \cdot (B\mathbf{z}) = 0, \quad (B\mathbf{z}) \cdot (B\mathbf{z}) = 1.$$

Thus $B\mathbf{y}$ must be a unit-length vector in $W_1^\perp$ and there must exist a real number θ such that

$$B\mathbf{y} = \cos\theta\mathbf{y} + \sin\theta\mathbf{z}.$$

Moreover, $B\mathbf{z}$ must be a unit-length vector in $W_1^\perp$ which is perpendicular to $B\mathbf{y}$ and hence

$$B\mathbf{z} = \pm(-\sin\theta\mathbf{y} + \cos\theta\mathbf{z}).$$

However, we cannot have

$$B\mathbf{z} = -(-\sin\theta\mathbf{y} + \cos\theta\mathbf{z}).$$

Indeed, this would imply (via a short calculation) that the vector

$$\mathbf{u} = \cos(\theta/2)\mathbf{y} + \sin(\theta/2)\mathbf{z}$$

is fixed by B, in other words $B\mathbf{u} = \mathbf{u}$, contradicting the fact that $\mathbf{u} \in W_1^\perp$. Thus we must have

$$B\mathbf{z} = -\sin\theta\mathbf{y} + \cos\theta\mathbf{z},$$

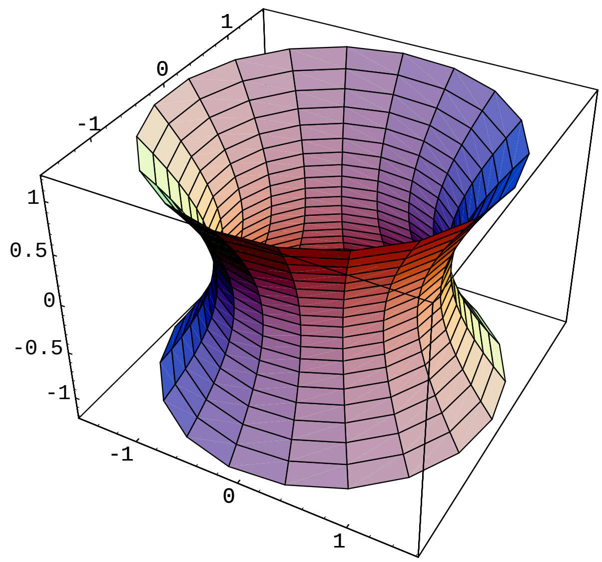

Figure 2.3: Hyperboloid of one sheet.

and multiplication by B must be a rotation in the plane $W_1^\perp$ through an angle θ. Moreover, it is easily checked that $\mathbf{y} + i\mathbf{z}$ and $\mathbf{y} - i\mathbf{z}$ are eigenvectors for B with eigenvalues

$$e^{\pm i\theta} = \cos\theta \pm i \sin\theta.$$

We can therefore conclude that a 3×3 orthogonal matrix B of determinant one represents a rotation about an axis (which is the eigenspace for eigenvalue one) and through an angle θ (which can be determined from the eigenvalues of B, which must be 1 and $e^{\pm i\theta}$).

Exercises:

2.2.1. Suppose that

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

a. Find an orthogonal matrix B such that $B^T A B$ is diagonal.

b. Sketch the conic section $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.

c. Sketch the conic section $2(x_1 - 1)^2 + 6(x_1 - 1)x_2 + 2x_2^2 = 1$.

2.2.2. Suppose that

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}.$$

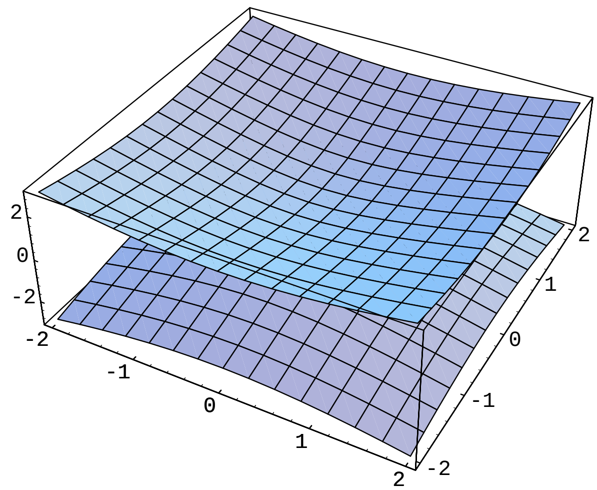

Figure 2.4: Hyperboloid of two sheets.

a. Find an orthogonal matrix B such that $B^T AB$ is diagonal.

b. Sketch the conic section $2x_1^2 - 4x_1x_2 + 5x_2^2 = 1$.

c. Sketch the conic section $2x_1^2 - 4x_1x_2 + 5x_2^2 - 4x_1 + 4x_2 = -1$.

2.2.3. Suppose that

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

a. Find an orthogonal matrix B such that $B^T AB$ is diagonal.

b. Sketch the conic section $4x_1^2 + 4x_1x_2 + x_2^2 - \sqrt{5}x_1 + 2\sqrt{5}x_2 = 0$.

2.2.4. Determine which of the following conic sections are ellipses, which are hyperbolas, etc.:

a. $x_1^2 + 4x_1x_2 + 3x_2^2 = 1$.

b. $x_1^2 + 6x_1x_2 + 10x_2^2 = 1$.

c. $-3x_1^2 + 6x_1x_2 - 4x_2^2 = 1$.

d. $-x_1^2 + 4x_1x_2 - 3x_2^2 = 1$.

2.2.5. Find the semi-major and semi-minor axes of the ellipse

$$5x_1^2 + 6x_1x_2 + 5x_2^2 = 4.$$

2.2.6. Suppose that

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

a. Find an orthogonal matrix B such that $B^T AB$ is diagonal.

b. Sketch the quadric surface $4x_1x_2 + 3x_2^2 + 9x_3^2 = 1$.

2.2.7. Determine which of the following quadric surfaces are ellipsoids, which are hyperboloids of one sheet, which are hyperboloids of two sheets, etc.:

a. $x_1^2 - x_2^2 - x_3^2 - 6x_2x_3 = 1$.

b. $x_1^2 + x_2^2 - x_3^2 - 6x_1x_2 = 1$.

c. $x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 2x_3 = 0$.

2.2.8. a. Show that the matrix

$$B = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

is an orthogonal matrix of deteminant one. Thus multiplication by B is rotation about some axis through some angle.

b. Find a nonzero vector which spans the axis of rotation.

c. Determine the angle of rotation.

2.3 Orthonormal bases

Recall that if $\mathbf{v}$ is an element of R^3, we can express it with respect to the standard basis as

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \quad \text{where} \quad a = \mathbf{v} \cdot \mathbf{i}, \quad b = \mathbf{v} \cdot \mathbf{j}, \quad c = \mathbf{v} \cdot \mathbf{k}.$$

We would like to extend this formula to the case where the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for R^3 is replaced by a more general "orthonormal basis."

Definition. A collection of n vectors $\mathbf{b}_1$, $\mathbf{b}_2$, ... , $\mathbf{b}_n$ in R^n is an *orthonormal basis* for R^n if

$$\begin{array}{cccc} \mathbf{b}_1 \cdot \mathbf{b}_1 = 1, & \mathbf{b}_1 \cdot \mathbf{b}_2 = 0, & \cdots, & \mathbf{b}_1 \cdot \mathbf{b}_n = 0, \\ & \mathbf{b}_2 \cdot \mathbf{b}_2 = 1, & \cdots, & \mathbf{b}_2 \cdot \mathbf{b}_n = 0, \\ & & & \cdot \\ & & & \mathbf{b}_n \cdot \mathbf{b}_n = 1. \end{array} \tag{2.9}$$

From the discussion in Section 2.1, we recognize that the term orthonormal basis is just another name for a collection of n vectors which form the columns of an orthogonal $n \times n$ matrix.

It is relatively easy to express an arbitrary vector $\mathbf{f} \in \mathsf{R}^n$ in terms of an orthonormal basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$: to find constants $c_1, c_2, \ldots, c_n$ so that

$$\mathbf{f} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n, \tag{2.10}$$

we simply dot both sides with the vector $\mathbf{b}_i$ and use (2.9) to conclude that

$$c_i = \mathbf{b}_i \cdot \mathbf{f}.$$

In other words, if $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ is an orthonormal basis for R^n, then

$$\mathbf{f} \in \mathsf{R}^n \quad \Rightarrow \quad \mathbf{f} = (\mathbf{f} \cdot \mathbf{b}_1)\mathbf{b}_1 + \cdots + (\mathbf{f} \cdot \mathbf{b}_n)\mathbf{b}_n, \tag{2.11}$$

a generalization of the formula we gave for expressing a vector in R^3 in terms of the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

This formula can be helpful in solving the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{f}, \tag{2.12}$$

in the case where A is a symmetric $n \times n$ matrix and $\mathbf{f}$ is a constant vector. Since A is symmetric, the Spectral Theorem of Section 2.1 guarantees that the eigenvalues of A are real and that we can find an $n \times n$ orthogonal matrix B such that

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

If we set $\mathbf{x} = B\mathbf{y}$, then

$$B\frac{d\mathbf{y}}{dt} = \frac{d\mathbf{x}}{dt} = A\mathbf{x} = AB\mathbf{y} \qquad \Rightarrow \qquad \frac{d\mathbf{y}}{dt} = B^{-1}AB\mathbf{y}.$$

Thus in terms of the new variable

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{pmatrix}, \qquad \text{we have} \qquad \begin{pmatrix} dy_1/dt \\ dy_2/dt \\ \cdot \\ dy_n/dt \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{pmatrix},$$

so that the matrix differential equation decouples into n noninteracting scalar differential equations

$$\begin{aligned} dy_1/dt &= \lambda_1 y_1, \\ dy_2/dt &= \lambda_2 y_2, \\ &\cdot \\ dy_n/dt &= \lambda_n y_n. \end{aligned}$$

We can solve these equations individually, obtaining the general solution

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \cdot \\ c_n e^{\lambda_n t} \end{pmatrix},$$

where $c_1, c_2, \ldots, c_n$ are constants of integration. Transferring back to our original variable $\mathbf{x}$ yields

$$\mathbf{x} = B \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \cdot \\ c_n e^{\lambda_n t} \end{pmatrix} = c_1 \mathbf{b}_1 e^{\lambda_1 t} + c_2 \mathbf{b}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{b}_n e^{\lambda_n t},$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ are the columns of B. Note that

$$\mathbf{x}(0) = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_n \mathbf{b}_n.$$

To finish solving the initial value problem (2.12), we need to determine the constants $c_1, c_2, \ldots, c_n$ so that

$$c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_n \mathbf{b}_n = \mathbf{f}.$$

It is here that our formula (2.11) comes in handy; using it we obtain

$$\mathbf{x} = (\mathbf{f} \cdot \mathbf{b}_1) \mathbf{b}_1 e^{\lambda_1 t} + \cdots + (\mathbf{f} \cdot \mathbf{b}_n) \mathbf{b}_n e^{\lambda_n t}.$$

Example. Suppose we want to solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{f}, \qquad \text{where} \qquad \mathbf{f} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}.$$

We saw in Section 2.1 that A has the eigenvalues $\lambda_1 = 1$ with multiplicity two and $\lambda_2 = 9$ with multiplicity one. Moreover, the orthogonal matrix

$$B = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}$$

has the property that

$$B^{-1}AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Thus the general solution to the matrix differential equation $d\mathbf{x}/dt = A\mathbf{x}$ is

$$\mathbf{x} = B \begin{pmatrix} c_1 e^t \\ c_2 e^t \\ c_3 e^{9t} \end{pmatrix} = c_1 \mathbf{b}_1 e^t + c_2 \mathbf{b}_2 e^t + c_3 \mathbf{b}_3 e^{9t},$$

where

$$\mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Setting $t = 0$ in our expression for $\mathbf{x}$ yields

$$\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3.$$

To solve the initial value problem, we employ (2.11) to see that

$$c_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 4, \qquad c_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \sqrt{2},$$

$$c_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 2\sqrt{2}.$$

Hence the solution is

$$\mathbf{x} = 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t + \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} e^t + 2\sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} e^{9t}.$$

Exercise:

2.3.1.a. Find the eigenvalues of the symmetric matrix

$$A = \begin{pmatrix} 5 & 4 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

b. Find an orthogonal matrix B such that $B^{-1}AB$ is diagonal.

c. Find an orthonormal basis for R^4 consisting of eigenvectors of A.

d. Find the general solution to the matrix differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

e. Find the solution to the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}.$$

2.3.2.a. Find the eigenvalues of the symmetric matrix

$$A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -5 \end{pmatrix}.$$

(Hint: To find roots of the cubic, try $\lambda = 1$ and $\lambda = -1$.)

b. Find an orthogonal matrix B such that $B^{-1}AB$ is diagonal.

c. Find an orthonormal basis for R^3 consisting of eigenvectors of A.

d. Find the general solution to the matrix differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

e. Find the solution to the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

2.3.3. (For students with access to Mathematica) a. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

by running the Mathematica program

```
a = {{2,1,1},{1,3,2},{1,2,4}}; Eigenvalues[a]
```

The answer is so complicated because Mathematica uses exact but complicated formulae for solving cubics discovered by Cardan and Tartaglia, two sixteenth century Italian mathematicians.

b. Find numerical approximations to these eigenvalues by running the program

```
Eigenvalues[N[a]]
```

The numerical values of the eigenvalues are far easier to use.

c. Use Mathematica to find numerical values for the eigenvectors for A by running the Mathematica program

```
Eigenvectors[N[a]]
```

and write down the general solution to the matrix differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

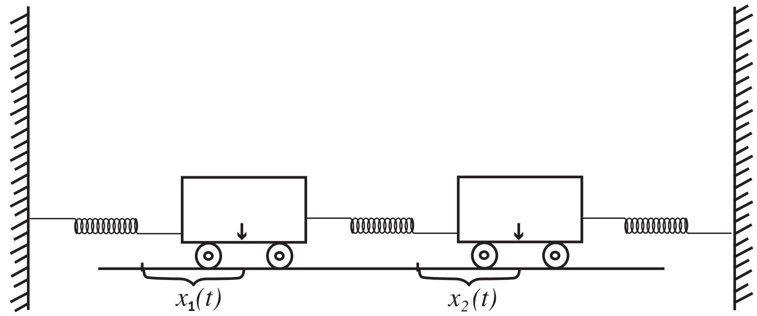

Figure 2.5: Two carts connected by springs and moving along a friction-free track.

2.4 Mechanical systems

Mechanical systems consisting of weights and springs connected together in an array often lead to initial value problems of the type

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{f}, \qquad \frac{d\mathbf{x}}{dt} = \mathbf{0}, \tag{2.13}$$

where A is a symmetric matrix and $\mathbf{f}$ is a constant vector. These can be solved by a technique similar to that used in the previous section.

For example, let us consider the mechanical system illustrated in Figure 2.5. Here we have two carts moving along a friction-free track, each containing mass m and attached together by three springs, with spring constants k_1, k_2 and k_3. Let

$$\begin{aligned} x_1(t) &= \text{the position of the first cart to the right of equilibrium,} \\ x_2(t) &= \text{the position of the second cart to the right of equilibrium,} \\ F_1 &= \text{force acting on the first cart,} \\ F_2 &= \text{force acting on the second cart,} \end{aligned}$$

with positive values for F_1 or F_2 indicating that the forces are pulling to the right, negative values that the forces are pulling to the left.

Suppose that when the carts are in equilibrium, the springs are also in equilibrium and exert no forces on the carts. In this simple case, it is possible to reason directly from Hooke's law that the forces F_1 and F_2 must be given by the formulae

$$F_1 = -k_1 x_1 + k_2(x_2 - x_1), \qquad F_2 = k_2(x_1 - x_2) - k_3 x_2,$$

but it becomes difficult to determine the forces for more complicated mechanical systems consisting of many weights and springs. Fortunately, there are some simple principles from physics which simply the procedure for finding the forces acting in such mechanical systems.

The easiest calculation of the forces is based upon the notion of *work*. On the one hand, the work required to pull a weight to a new position is equal to the increase in potential energy imparted to the weight. On the other hand, we have the equation

$$\text{Work} = \text{Force} \times \text{Displacement},$$

which implies that

$$\text{Force} = \frac{\text{Work}}{\text{Displacement}} = -\frac{\text{Change in potential energy}}{\text{Displacement}}.$$

Thus if $V(x_1, x_2)$ is the potential energy of the configuration when the first cart is located at the point x_1 and the second cart is located at the point x_2, then the forces are given by the formulae

$$F_1 = -\frac{\partial V}{\partial x_1}, \qquad F_2 = -\frac{\partial V}{\partial x_2}.$$

In our case, the potential energy V is the sum of the potential energies stored in each of the three springs,

$$V(x_1, x_2) = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3 x_2^2,$$

and hence we obtain the formulae claimed before:

$$F_1 = -\frac{\partial V}{\partial x_1} = -k_1 x_1 + k_2(x_2 - x_1),$$

$$F_2 = -\frac{\partial V}{\partial x_2} = k_2(x_1 - x_2) - k_3 x_2.$$

It now follows from Newton's second law of motion that

$$\text{Force} = \text{Mass} \times \text{Acceleration},$$

and hence

$$F_1 = m\frac{d^2 x_1}{dt^2}, \qquad F_2 = m\frac{d^2 x_2}{dt^2}.$$

Thus we obtain a second-order system of differential equations,

$$m\frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2,$$

$$m\frac{d^2 x_2}{dt^2} = k_2(x_1 - x_2) - k_3 x_2 = k_2 x_1 - (k_2 + k_3)x_2.$$

We can write this system in matrix form as

$$m\frac{d^2 \mathbf{x}}{dt^2} = A\mathbf{x}, \qquad \text{where} \qquad A = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{pmatrix}. \tag{2.14}$$

Note that A is indeed a symmetric matrix. The potential energy is given by the expression

$$V(x_1, x_2) = -\frac{1}{2}\,(x_1 x_2)\, A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Example. Let us consider the special case of the mass-spring system in which

$$m = k_1 = k_2 = k_3 = 1,$$

so that the system (2.14) becomes

$$\frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}. \tag{2.15}$$

To find the eigenvalues, we must solve the characteristic equation

$$\det \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (\lambda + 2)^2 - 1 = 0,$$

which yields

$$\lambda = -2 \pm 1.$$

The eigenvalues in this case are

$$\lambda_1 = -1, \quad \text{and} \quad \lambda_2 = -3.$$

The eigenspace corresponding to the eigenvalue -1 is

$$W_{-1} = \{\mathbf{b} \in R^2 : (A + I)\mathbf{b} = 0\} = \ldots = \operatorname{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It follows from the argument in Section 2.1 that the eigenspace corresponding to the other eigenvalue is just the orthogonal complement

$$W_{-3} = \operatorname{span} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Unit length eigenvectors lying in the two eigenspaces are

$$\mathbf{b}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

The theorem of Section 2.1 guarantees that the matrix

$$B = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

whose columns are $\mathbf{b}_1$ and $\mathbf{b}_2$, will diagonalize our system of differential equations.

Indeed, if we define new coordinates (y_1, y_2) by setting

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

our system of differential equations transforms to

$$\begin{aligned} d^2y_1/dt^2 &= -y_1, \\ d^2y_2/dt^2 &= -3y_2. \end{aligned}$$

We set $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$, so that this system assumes the familiar form

$$\begin{aligned} d^2y_1/dt^2 + \omega_1^2 y_1 &= 0, \\ d^2y_2/dt^2 + \omega_2^2 y_2 &= 0, \end{aligned}$$

a system of two noninteracting harmonic oscillators.

The general solution to the transformed system is

$$y_1 = a_1 \cos \omega_1 t + b_1 \sin \omega_1 t, \qquad y_2 = a_2 \cos \omega_2 t + b_2 \sin \omega_2 t.$$

In the original coordinates, the general solution to (2.15) is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \end{pmatrix},$$

or equivalently,

$$\mathbf{x} = \mathbf{b}_1(a_1 \cos \omega_1 t + b_1 \sin \omega_1 t) + \mathbf{b}_2(a_2 \cos \omega_2 t + b_2 \sin \omega_2 t).$$

The motion of the carts can be described as a general superposition of two *modes* of oscillation, of frequencies

$$\frac{\omega_1}{2\pi} \qquad \text{and} \qquad \frac{\omega_2}{2\pi}.$$

Exercises:

2.4.1.a. Consider the mass-spring system with two carts illustrated in Figure 2.5 in the case where $k_1 = 4$ and $m = k_2 = k_3 = 1$. Write down a system of second-order differential equations which describes the motion of this system.

b. Find the general solution to this system.

c. What are the frequencies of vibration of this mass-spring system?

2.4.2.a. Consider the mass-spring system with three carts illustrated in Figure 2.5 in the case where $m = k_1 = k_2 = k_3 = k_4 = 1$. Show that the motion of this system is described by the matrix differential equation

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}, \qquad \text{where} \qquad A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

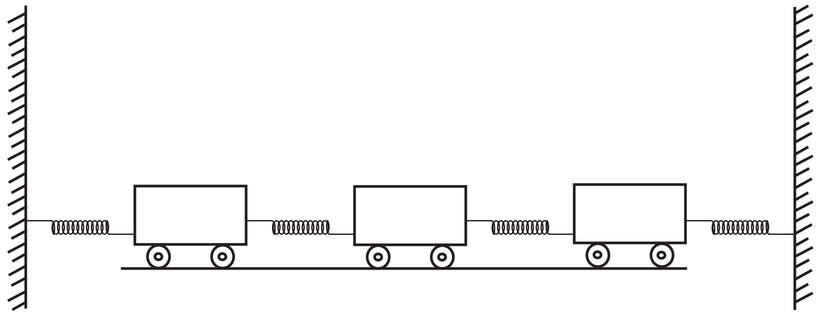

Figure 2.6: Three carts connected by springs and moving along a friction-free track.

b. Find the eigenvalues of the symmetric matrix A.

c. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

d. Find an orthogonal matrix B such that $B^{-1}AB$ is diagonal.

e. Find the general solution to the matrix differential equation

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}.$$

f. Find the solution to the initial value problem

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \frac{d\mathbf{x}}{dt}(0) = \mathbf{0}.$$

2.4.3.a. Find the eigenvalues of the symmetric matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

b. What are the frequencies of oscillation of a mechanical system which is governed by the matrix differential equation

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}?$$

2.5 Mechanical systems with many degrees of freedom*

Using an approach similar to the one used in the preceding section, we can consider more complicated systems consisting of many masses and springs. For

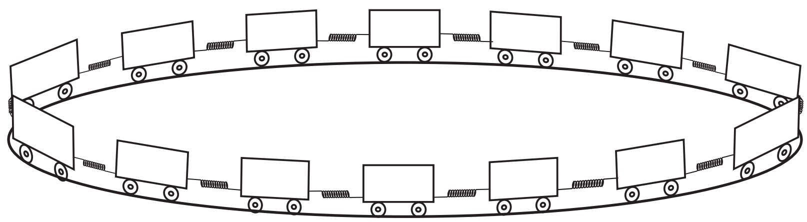

Figure 2.7: A circular array of carts and springs.

example, we could consider the box spring underlying the mattress in a bed. Although such a box spring contains hundreds of individual springs, and hence the matrix A in the corresponding dynamical system constains hundreds of rows and columns, it is still possible to use symmetries in the box spring to simplify the calculations, and make the problem of determining the "natural frequencies of vibration" of the mattress into a manageable problem.

To illustrate how the symmetries of a problem can make it much easier to solve, let us consider a somewhat simpler problem, a system of n carts containing identical weights of mass m and connected by identical springs of spring constant k, moving along a circular friction-free track as sketched in Figure 2.6.

We choose a positive direction along the track and let x_i denote the displacement of the i-th cart out of equilibrium position in the positive direction, for $1 \leq i \leq n$. The potential energy stored in the springs is

$$V(x_1, \ldots, x_n) = \frac{1}{2}k(x_n - x_1)^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \ldots + \frac{1}{2}k(x_n - x_{n-1})^2$$

$$= -\frac{1}{2}k \begin{pmatrix} x_1 & x_2 & x_3 & \cdot & x_n \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ x_n \end{pmatrix}.$$

We can write this as

$$V(x_1, \ldots, x_n) = -\frac{1}{2}k\mathbf{x}^T A\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & -2 \end{pmatrix},$$

or equivalently as

$$V(x_1, \ldots, x_n) = -\frac{1}{2}k \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j,$$

where a_{ij} denotes the (i,j)-component of the matrix A.

Just as in the preceding section, the force acting on the i-th cart can be calculated as minus the derivative of the potential energy with respect to the position of the i-th cart, the other carts being held fixed. Thus for example,

$$F_1 = -\frac{\partial V}{\partial x_1} = \frac{1}{2}k\sum_{j=1}^{n} a_{1j}x_j + \frac{1}{2}k\sum_{i=1}^{n} a_{i1}x_i = k\sum_{j=1}^{n} a_{1j}x_j,$$

the last step obtained by using the symmetry of A. In general, we obtain the result:

$$F_i = -\frac{\partial V}{\partial x_i} = k\sum_{j=1}^{n} a_{ij}x_j,$$

which could be rewritten in matrix form as

$$\mathbf{F} = kA\mathbf{x}. \tag{2.16}$$

On the other hand, by Newton's second law of motion,

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}.$$

Substitution into (2.16) yields

$$m\frac{d^2\mathbf{x}}{dt^2} = kA\mathbf{x} \quad \text{or} \quad \frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}A\mathbf{x}, \tag{2.17}$$

where A is the symmetric matrix given above. To find the frequencies of vibration of our mechanical system, we need to find the eigenvalues of the $n \times n$ matrix A.

To simplify the calculation of the eigenvalues of this matrix, we make use of the fact that the carts are identically situated—if we relabel the carts, shifting them to the right by one, the dynamical system remains unchanged. Indeed, lew us define new coordinates $(y_1, \ldots, y_n)$ by setting

$$x_1 = y_2, \quad x_2 = y_3, \quad \ldots, \quad x_{n-1} = y_n, \quad x_n = y_1,$$

or in matrix terms,

$$\mathbf{x} = T\mathbf{y}, \quad \text{where} \quad T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\mathbf{y}$ satisfies exactly the same system of differential equations as $\mathbf{x}$:

$$\frac{d^2\mathbf{y}}{dt^2} = \frac{k}{m}A\mathbf{y}. \tag{2.18}$$

On the other hand,

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}A\mathbf{x} \qquad \Rightarrow \qquad T\frac{d^2\mathbf{y}}{dt^2} = \frac{k}{m}AT\mathbf{y} \qquad \text{or} \qquad \frac{d^2\mathbf{y}}{dt^2} = \frac{k}{m}T^{-1}AT\mathbf{y}.$$

Comparison with (2.18) yields

$$A = T^{-1}AT, \quad \text{or} \quad TA = AT.$$

In other words the matrices A and T *commute*.

Now it is quite easy to solve the eigenvector-eigenvalue problem for T. Indeed, if $\mathbf{x}$ is an eigenvector for T corresponding to the eigenvalue λ, the components of $\mathbf{x}$ must satisfy the vector equation $T\mathbf{x} = \lambda\mathbf{x}$. In terms of components, the vector equation becomes

$$x_2 = \lambda x_1, \quad x_3 = \lambda x_2, \quad \ldots, x_n = \lambda x_{n-1}, \quad x_1 = \lambda x_n. \tag{2.19}$$

Thus $x_3 = \lambda^2 x_1$, $x_4 = \lambda^3 x_1$, and so forth, yielding finally the equation

$$x_1 = \lambda^n x_1.$$

Similarly,

$$x_2 = \lambda^n x_2, \quad \ldots, \quad x_n = \lambda^n x_n.$$

Since at least one of the x_i's is nonzero, we must have

$$\lambda^n = 1. \tag{2.20}$$

This equation is easily solved via Euler's formula:

$$1 = e^{2\pi i} \Rightarrow (e^{2\pi i/n})^n = 1, \quad \text{and similarly} \quad [(e^{2\pi i/n})^j]^n = 1,$$

for $0 \leq j \leq n-1$. Thus the solutions to (2.20) are

$$\lambda = \eta^j, \quad \text{for } 0 \leq j \leq n-1, \quad \text{where} \quad \eta = e^{2\pi i/n}. \tag{2.21}$$

For each choice of j, we can try to find eigenvectors corresponding to η^j. If we set $x_1 = 1$, we can conclude from (2.19) that

$$x_2 = \eta^j, \quad x_3 = \eta^{2j}, \quad \ldots, \quad x_n = \eta^{(n-1)j},$$

thereby obtaining a nonzero solution to the eigenvector equation. Thus for each j, $0 \leq j \leq n-1$, we do indeed have an eigenvector

$$\mathbf{e}_j = \begin{pmatrix} 1 \\ \eta^j \\ \eta^{2j} \\ \cdot \\ \eta^{(n-1)j} \end{pmatrix},$$

for the eigenvalue η^j. Moreover, each eigenspace is one-dimensional.

In the dynamical system that we are considering, of course, we need to solve the eigenvalue-eigenvector problem for A, not for T. Fortunately, however, since A and T commute, the eigenvectors for T are also eigenvectors for A. Indeed, since $AT = TA$,

$$T(A\mathbf{e}_j) = A(T\mathbf{e}_j) = A(\eta^j\mathbf{e}_j) = \eta^j(A\mathbf{e}_j),$$

and this equation states that $A\mathbf{e}_j$ is an eigenvector for T with the same eigenvalue as $\mathbf{e}_j$. Since the eigenspaces of T are all one-dimensional, $A\mathbf{e}_j$ must be a multiple of $\mathbf{e}_j$; in other words,

$$A\mathbf{e}_j = \lambda_j\mathbf{e}_j, \quad \text{for some number } \lambda_j.$$

To find the eigenvalues λ_j for A, we simply act on $\mathbf{e}_j$ by A: we find that the first component of the vector $A\mathbf{e}_j = \lambda_j\mathbf{e}_j$ is

$$-2 + \eta^j + \eta^{(n-1)j} = -2 + (e^{2\pi ij/n} + e^{-2\pi ij/n}) = -2 + 2\cos(2\pi j/n),$$

where we have used Euler's formula once again. On the other hand, the first component of $\mathbf{e}_j$ is 1, so we immediately conclude that

$$\lambda_j = -2 + 2\cos(2\pi j/n).$$

It follows from the familiar formula

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

that

$$\lambda_j = -2 + 2[\cos(\pi j/n)]^2 - 2[\sin(\pi j/n)]^2 = -4[\sin(\pi j/n)]^2.$$

Note that $\lambda_j = \lambda_{n-j}$, and hence the eigenspaces for A are two-dimensional, except for the eigenspace corresponding to $j = 0$, and if n is even, to the eigenspace corresponding to $j = n/2$. Thus in the special case where n is odd, all the eigenspaces are two-dimensional except for the eigenspace with eigenvalue $\lambda_0 = 0$, which is one-dimensional.

If $j \neq 0$ and $j \neq n/2$, $\mathbf{e}_j$ and $\mathbf{e}_{n-j}$ form a basis for the eigenspace corresponding to eigenvalue λ_j. It is not difficult to verify that

$$\frac{1}{2}(\mathbf{e}_j + \mathbf{e}_{n-j}) = \begin{pmatrix} 1 \\ \cos(\pi j/n) \\ \cos(2\pi j/n) \\ \cdot \\ \cos((n-1)\pi j/n) \end{pmatrix}$$

and

$$\frac{i}{2}(\mathbf{e}_j - \mathbf{e}_{n-j}) = \begin{pmatrix} 0 \\ \sin(\pi j/n) \\ \sin(2\pi j/n) \\ \cdot \\ \sin((n-1)\pi j/n) \end{pmatrix}$$

form a real basis for this eigenspace. Let

$$\omega_j = \sqrt{-\lambda_j} = 2\sin(\pi j/n).$$

In the case where n is odd, we can write the general solution to our dynamical system (2.17) as

$$\mathbf{x} = \mathbf{e}_0(a_0 + b_0 t) + \sum_{j=1}^{(n-1)/2} \frac{1}{2}(\mathbf{e}_j + \mathbf{e}_{n-j})(a_j \cos\omega_j t + b_j \sin\omega_j t)$$

$$+ \sum_{j=1}^{(n-1)/2} \frac{i}{2}(\mathbf{e}_j - \mathbf{e}_{n-j})(c_j \cos\omega_j t + d_j \sin\omega_j t).$$

The motion of the system can be described as a superposition of several modes of oscillation, the frequencies of oscillation being

$$\frac{\omega_j}{2\pi} = \sqrt{\frac{k}{m}}\frac{\sin(\pi j/n)}{\pi}.$$

Note that the component

$$\mathbf{e}_0(a_0 + b_0 t) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ 1 \end{pmatrix} (a_0 + b_0 t)$$

corresponds to a constant speed motion of the carts around the track. If n is large,

$$\frac{\sin(\pi/n)}{\pi} \doteq \frac{\pi/n}{\pi} = \frac{1}{n} \qquad \Rightarrow \qquad \frac{\omega_1}{2\pi} \doteq \sqrt{\frac{k}{m}}\frac{1}{n},$$

so if we were to set $k/m = n^2$, the lowest nonzero frequency of oscillation would approach one as $n \to \infty$.

Exercise:

2.5.1. Find the eigenvalues of the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

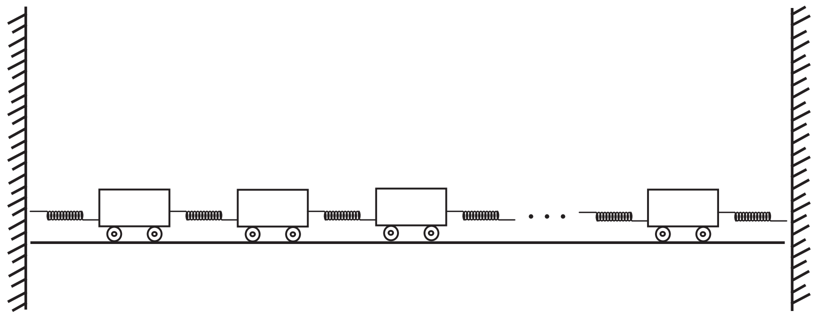

Figure 2.8: A linear array of carts and springs.

by expanding the determinant

$$\begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ 0 & 0 & -\lambda & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & -\lambda \end{vmatrix}.$$

2.6 A linear array of weights and springs*

Suppose more generally that a system of $n-1$ carts containing identical weights of mass m, and connected by identical springs of spring constant k, are moving along a friction-free track, as shown in Figure 2.7. Just as in the preceding section, we can show that the carts will move in accordance with the linear system of differential equations

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x} = \frac{k}{m}P\mathbf{x},$$

where

$$P = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -2 \end{pmatrix}.$$

We take $k = n$ and $m = (1/n)$, so that $k/m = n^2$.

We can use the following Mathematica program to find the eigenvalues of the $n \times n$ matrix A, when $n = 6$:

```
n = 6;
m = Table[Max[2-Abs[i-j],0], {i,n-1} ,{j,n-1} ];
p = m - 4 IdentityMatrix[n-1];
```

```
a = n∧2 p
eval = Eigenvalues[N[a]]
```

Since

$$2 - |i-j| \begin{cases} = 2 & \text{if } j = i, \\ = 1 & \text{if } j = i \pm 1 \\ \leq 0 & \text{otherwise,} \end{cases}$$

the first line of the program generates the $(n-1) \times (n-1)$-matrix

$$M = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & 2 \end{pmatrix}.$$

The next two lines generate the matrices P and $A = n^2 P$. Finally, the last line finds the eigenvalues of A. If you run this program, you will note that all the eigenvalues are negative and that Mathematica provides the eigenvalues in increasing order.

We can also modify the program to find the eigenvalues of the matrix when $n = 12$, $n = 26$, or $n = 60$ by simply replacing 6 in the top line with the new value of n. We can further modify the program by asking for only the smallest eigenvalue `lambda[[n]]` and a plot of the corresponding eigenvector:

```
n = 14;
m = Table[Max[2-Abs[i-j],0], {i,n-1} ,{j,n-1} ];
p = m - 4 IdentityMatrix[n-1]; a = n∧2 p;
eval = Eigenvalues[N[a]]; evec = Eigenvectors[N[a]];
Print[eval[[n-1]]];
ListPlot[evec[[n-1]]];
```

If we experiment with this program using different values for n, we will see that as n gets larger, the smallest eigenvalue seems to approach $-\pi^2$ and the plot of the smallest eigenvector looks more and more like a sine curve. Thus the fundamental frequency of the mechanical system seems to approach $\pi/2\pi = 1/2$ and the oscillation in the fundamental mode appears more and more sinusoidal in shape.

When n is large, we can consider this array of springs and weights as a model for a string situated along the x-axis and free to stretch to the right and the left along the x-axis. The track on which the cart runs restricts the motion of the weights to be only in the x-direction. A more realistic model would allow the carts to move in all three directions. This would require new variables y_i and z_i such that

$$y_i(t) = \text{the } y\text{-component of displacement of the } i\text{-th weight,}$$

$$z_i(t) = \text{the } z\text{-component of displacement of the } i\text{-th weight.}$$

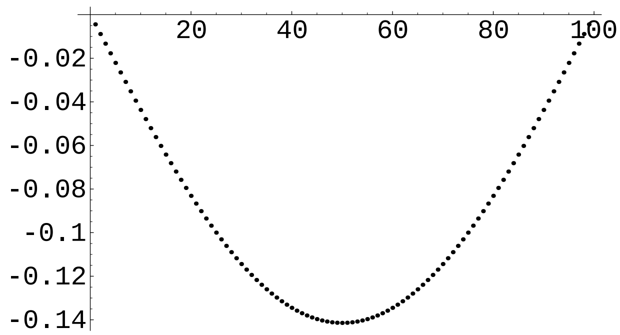

Figure 2.9: Shape of the fundamental mode (n=100).

If we were to set

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ z_n \end{pmatrix},$$

then an argument similar to the one given above would show that the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ would satisfy the matrix differential equations

$$\frac{d\mathbf{x}}{dt} = \frac{k}{m} A\mathbf{x}, \quad \frac{d\mathbf{y}}{dt} = \frac{k}{m} A\mathbf{y}, \quad \frac{d\mathbf{z}}{dt} = \frac{k}{m} A\mathbf{z}.$$

and each of these could be solved just as before.

Using a similar approach, we could consider even more complicated systems consisting of many masses and springs connected in a two- or three-dimensional array.

Chapter 3

Fourier Series

3.1 Fourier series

The theory of Fourier series and the Fourier transform is concerned with dividing a function into a superposition of sines and cosines, its components of various frequencies. It is a crucial tool for understanding waves, including water waves, sound waves and light waves. Suppose, for example, that the function $f(t)$ represents the amplitude of a light wave arriving from a distant galaxy. The light is a superposition of many frequencies which encode information regarding the material which makes up the stars of the galaxy, the speed with which the galaxy is receding from the earth, its speed of rotation, and so forth. Much of our knowledge of the universe is derived from analyzing the spectra of stars and galaxies. Just as a prism or a spectrograph is an experimental apparatus for dividing light into its components of various frequencies, so Fourier analysis is a mathematical technique which enables us to decompose an arbitrary function into a superposition of oscillations.

In the following chapter, we will describe how the theory of Fourier series can be used to analyze the flow of heat in a bar and the motion of a vibrating string. Indeed, Joseph Fourier's original investigations which led to the theory of Fourier series were motivated by an attempt to understand heat flow.[1] Nowadays, the notion of dividing a function into its components with respect to an appropriate "orthonormal basis of functions" is one of the key ideas of applied mathematics, useful not only as a tool for solving partial differential equations, as we will see in the next two chapters, but for many other purposes as well. For example, a black and white photograph could be represented by a function $f(x, y)$ of two variables, $f(x, y)$ representing the darkness at the point (x, y). The photograph can be stored efficiently by determining the components of $f(x, y)$ with respect to a well-chosen "wavelet basis." This idea is the key to image compression, which can be used to send pictures quickly over the internet.[2]

[1] Fourier's research was published in his *Théorie analytique de la chaleur* in 1822.

[2] See Stéphane Mallat, *A wavelet tour of signal processing*, Academic Press, Boston, 1998.

We turn now to the basics of Fourier analysis in its simplest context. A function $f : \mathsf{R} \to \mathsf{R}$ is said to be *periodic* of period T if it satisfies the relation

$$f(t+T) = f(t), \quad \text{for all } t \in \mathsf{R}.$$

Thus $f(t) = \sin t$ is periodic of period 2π.

Given an arbitrary period T, it is easy to construct examples of functions which are periodic of period T—indeed, the function $f(t) = \sin(\frac{2\pi t}{T})$ is periodic of period T because

$$\sin(\frac{2\pi(t+T)}{T}) = \sin(\frac{2\pi t}{T} + 2\pi) = \sin(\frac{2\pi t}{T}).$$

More generally, if k is any positive integer, the functions

$$\cos(\frac{2\pi kt}{T}) \quad \text{and} \quad \sin(\frac{2\pi kt}{T})$$

are also periodic functions of period T.

The main theorem from the theory of Fourier series states that any "well-behaved" periodic function of period T can be expressed as a superposition of sines and cosines:

$$f(t) = \frac{a_0}{2} + a_1\cos(\frac{2\pi t}{T}) + a_2\cos(\frac{4\pi t}{T}) + \ldots + b_1\sin(\frac{2\pi t}{T}) + b_2\sin(\frac{4\pi t}{T}) + \ldots . \tag{3.1}$$

In this formula, the a_k's and b_k's are called the *Fourier coefficients* of f, and the infinite series on the right-hand side is called the *Fourier series* of f.

Our first goal is to determine how to calculate these Fourier coefficients. For simplicity, we will restrict our attention to the case where the period $T = 2\pi$, so that

$$f(t) = \frac{a_0}{2} + a_1\cos t + a_2\cos 2t + \ldots + b_1\sin t + b_2\sin 2t + \ldots . \tag{3.2}$$

The formulae for a general period T are only a little more complicated, and are based upon exactly the same ideas.

The coefficient a_0 is particularly easy to evaluate. We simply integrate both sides of (3.2) from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} \frac{a_0}{2}dt + \int_{-\pi}^{\pi} a_1\cos tdt + \int_{-\pi}^{\pi} a_2\cos 2tdt + \ldots$$
$$+ \int_{-\pi}^{\pi} b_1\sin tdt + \int_{-\pi}^{\pi} b_2\sin 2tdt + \ldots .$$

Since the integral of $\cos kt$ or $\sin kt$ over the interval from $-\pi$ to π vanishes, we conclude that

$$\int_{-\pi}^{\pi} f(t)dt = \pi a_0,$$

and we can solve for a_0, obtaining

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt. \tag{3.3}$$

To find the other Fourier coefficients, we will need some integral formulae. We claim that if m and n are positive integers,

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases} \tag{3.4}$$

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases} \tag{3.5}$$

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0. \tag{3.6}$$

Let us verify the first of these equations. We will use the trigonometric identities

$$\cos((n+m)t) = \cos nt \cos mt - \sin nt \sin mt,$$

$$\cos((n-m)t) = \cos nt \cos mt + \sin nt \sin mt.$$

Adding these together, we obtain

$$\cos((n+m)t) + \cos((n-m)t) = 2\cos nt \cos mt,$$

or

$$\cos nt \cos mt = \frac{1}{2}(\cos((n+m)t) + \cos((n-m)t)).$$

Hence

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \frac{1}{2}\int_{-\pi}^{\pi} (\cos((n+m)t) + \cos((n-m)t))dt,$$

and since

$$\int_{-\pi}^{\pi} \cos(kt)dt = \frac{1}{k}\sin(kt)\bigg|_{-\pi}^{\pi} = 0,$$

if $k \neq 0$, we conclude that

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n. \end{cases}$$

The reader is asked to verify the other two integral formulae (3.5) and (3.6) in the exercises at the end of this section.

To find the formula for the Fourier coefficients a_k for $k > 0$, we multiply both sides of (3.2) by $\sin kt$ and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(t)\cos ktdt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos ktdt + \int_{-\pi}^{\pi} a_1 \cos t \cos ktdt + \dots$$
$$+ \dots + \int_{-\pi}^{\pi} a_k \cos kt \cos ktdt + \dots$$
$$+ \dots + \int_{-\pi}^{\pi} b_j \sin jt \cos ktdt + \dots .$$

According to our formulae (3.4) and (3.6), there is only one term which survives:

$$\int_{-\pi}^{\pi} f(t)\cos ktdt = \int_{-\pi}^{\pi} a_k \cos kt \cos ktdt = \pi a_k.$$

We can easily solve for a_k:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt. \tag{3.7}$$

A very similar argument yields the formula

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt. \tag{3.8}$$

Example. Let us use formulae (3.3), (3.7), and (3.8) to find the Fourier coefficients of the function

$$f(t) = \begin{cases} -\pi, & \text{for } -\pi < t < 0, \\ \pi, & \text{for } 0 < t < \pi, \\ 0, & \text{for } t = 0, \pi, \end{cases}$$

extended to be periodic of period 2π. Then

$$\frac{a_0}{2} = \text{average value of } f = 0,$$

and

$$a_m = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(t)\cos mtdt\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos mtdt\right] + \frac{1}{\pi}\left[\int_{0}^{\pi} \pi\cos mtdt\right]$$
$$= \frac{1}{\pi}\frac{-\pi}{m}[\sin mt]_{-\pi}^{0} + \frac{1}{\pi}\frac{\pi}{m}[\sin mt]_{0}^{\pi} = \cdots = 0,$$

while

$$b_m = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(t)\sin mtdt\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin mtdt\right] + \frac{1}{\pi}\left[\int_{0}^{\pi} \pi\sin mtdt\right]$$

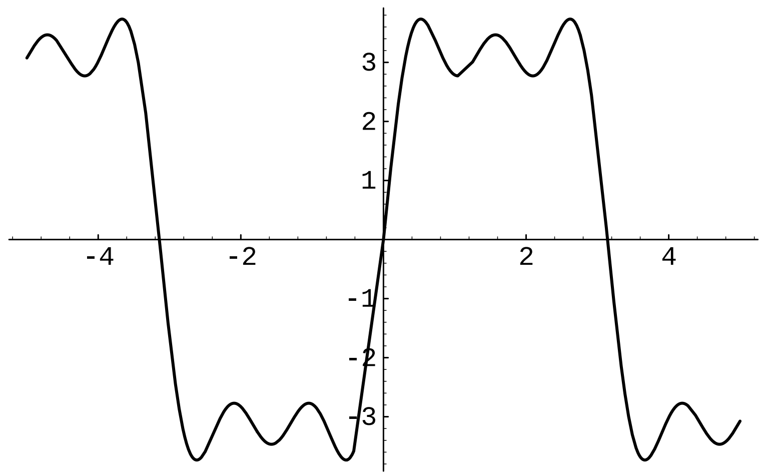

Figure 3.1: A graph of the Fourier approximation $\phi_5(t) = 4\sin t + (4/3)\sin 3t + (4/5)\sin 5t$.

$$= \frac{1}{\pi}\frac{\pi}{m}[\cos mt]_{-\pi}^{0} + \frac{1}{\pi}\frac{-\pi}{m}[\cos mt]_0^{\pi} = \frac{2}{m} - \frac{2}{m}\cos m\pi$$

$$= \frac{2}{m}(1-(-1)^m) = \begin{cases} \frac{4}{m}, & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases}$$

Thus we find the Fourier series for f:

$$f(t) = 4\sin t + \frac{4}{3}\sin 3t + \frac{4}{5}\sin 5t + \cdots .$$

The trigonometric polynomials

$$\phi_1(t) = 4\sin t, \quad \phi_3(t) = 4\sin t + \frac{4}{3}\sin 3t$$

$$\phi_5(t) = 4\sin t + \frac{4}{3}\sin 3t + \frac{4}{5}\sin 5t$$

are approximations to $f(t)$ which improve as the number of terms increases.

By the way, this Fourier series yields a curious formula for π. If we set $x = \pi/2$, $f(x) = \pi$, and we obtain

$$\pi = 4\sin(\pi/2) + \frac{4}{3}\sin(3\pi/2) + \frac{4}{5}\sin(5\pi/2) + \cdots$$

from which we conclude that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right).$$

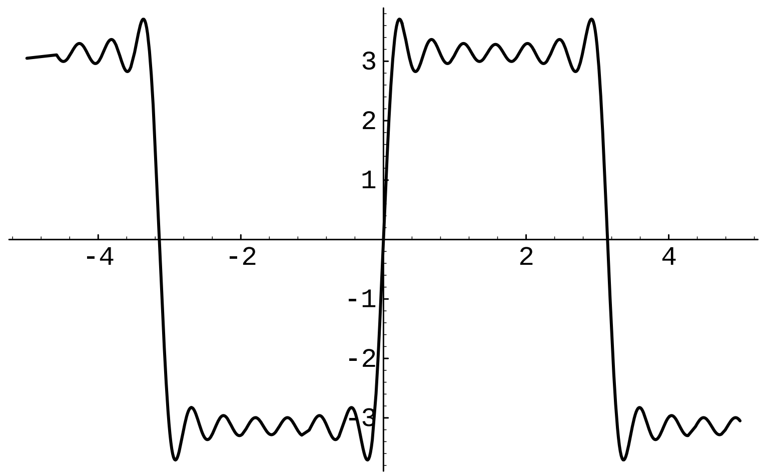

Figure 3.2: A graph of the Fourier approximation $\phi_{13}(t)$. The overshooting near the points of discontinuity is known as the "Gibbs phenomenon."

The Fourier series on the right-hand side of (3.1) is often conveniently expressed in the Σ notation,

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi kt}{T}) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi kt}{T}), \tag{3.9}$$

just as we did for power series in Chapter 1.

It is an interesting and difficult problem in harmonic analysis to determine how "well-behaved" a periodic function $f(t)$ must be in order to ensure that it can be expressed as a superposition of sines and cosines. An easily stated theorem, sufficient for many applications is:

Theorem 1. *If f is a continuous periodic function of period T, with condinuous derivative f', then f can be written uniquely as a superposition of sines and cosines, in accordance with (3.9), where the a_k's and b_k's are constants. Moreover, the infinite series on the right-hand side of (3.9) converges to $f(t)$ for every choice of t.*

However, often one wants to apply the theory of Fourier series to functions which are not quite so well-behaved, in fact to functions that are not even be continuous, such as the function in our previous example. A weaker sufficient condition for f to possess a Fourier series is that it be piecewise smooth.

The technical definition goes like this: A function $f(t)$ which is periodic of period T is said to be *piecewise smooth* if it is continuous and has a continuous derivative $f'(t)$ except at finitely many points of discontinuity within the interval $[0, T]$, and at each point t_0 of discontinuity, the right- and left-handed limits of

f and f',

$$\lim_{t \to t_0+}(f(t)), \quad \lim_{t \to t_0-}(f(t)), \qquad \lim_{t \to t_0+}(f'(t)), \quad \lim_{t \to t_0-}(f'(t)),$$

all exist. The following theorem, proven in more advanced books,[3] ensures that a Fourier decomposition can be found for any function which is piecewise smooth:

Theorem 2. *If f is any piecewise smooth periodic function of period T, f can be expressed as a Fourier series,*

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi kt}{T}) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi kt}{T})$$

where the a_k's and b_k's are constants. Here equality means that the infinite sum on the right converges to $f(t)$ for each t at which f is continuous. If f is discontinuous at t_0, its Fourier series at t_0 will converge to the average of the right and left hand limits of f as $t \to t_0$.

Exercises:

3.1.1. The function $f(t) = \cos^2 t$ can be regarded as either periodic of period π or periodic of period 2π. Choose one of the two periods and find the Fourier series of $f(t)$. (Hint: This problem is very easy if you use trigonometric identities instead of trying to integrate directly.)

3.1.2. The function $f(t) = \sin^3 t$ is periodic of period 2π. Find its Fourier series.

3.1.3. The function

$$f(t) = \begin{cases} t, & \text{for } -\pi < t < \pi, \\ 0, & \text{for } t = \pi, \end{cases}$$

can be extended to be periodic of period 2π. Find the Fourier series of this extension.

3.1.4. The function

$$f(t) = |t|, \qquad \text{for } t \in [-\pi, \pi],$$

can be extended to be periodic of period 2π. Find the Fourier series of this extension.

3.1.5. Find the Fourier series of the following function:

$$f(t) = \begin{cases} t^2, & \text{for } -\pi \leq t < \pi, \\ f(t - 2k\pi), & \text{for } -\pi + 2k\pi \leq t < \pi + 2k\pi. \end{cases}$$

3.1.6. Establish the formulae (3.5) and (3.6), which were given in the text.

[3] See, for example, Ruel Churchill and James Brown, *Fourier series and boundary value problems*, 4th edition, McGraw-Hill, New York, 1987 or Robert Seeley, *An introduction to Fourier series and integrals*, Benjamin, New York, 1966.

3.2 Inner products

There is a convenient way of remembering the formulae for the Fourier coefficients that we derived in the preceding section. Let V be the set of piecewise smooth functions which are periodic of period 2π. We say that V is a *vector space* because elements of V can be added and multiplied by scalars, these operations satisfying the same rules as those for addition of ordinary vectors and multiplication of ordinary vectors by scalars. We define an "inner product" between elements of V by means of the formula

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt.$$

Thus for example, if $f(t) = \sin t$ and $g(t) = 2\cos t$, then

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 2\sin t\cos t dt = \int_{-\pi}^{\pi} \sin(2t)dt = -\cos(2t)|_{-\pi}^{\pi} = 0.$$

The remarkable fact is that this inner product has properties quite similar to those of the standard dot product on R^n:

- $\langle f, g\rangle = \langle g, f\rangle$, whenever f and g are elements of V.
- $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$.
- $\langle cf, g\rangle = c\langle f, g\rangle$, when c is a real constant.
- $\langle f, f\rangle \geq 0$, with equality holding only if $f = 0$ (at all points of continuity).

This suggests that we might use geometric terminology for elements of V just as we did for vectors in R^n. Thus, for example, we will say that an element f of V is of *unit length* if $\langle f, f\rangle = 1$ and that two elements f and g of V are *perpendicular* if $\langle f, g\rangle = 0$.

In this terminology, the formulae (3.4), (3.5), and (3.6) can be expressed by stating that the functions

$$1/\sqrt{2}, \quad \cos t, \quad \cos 2t, \quad \cos 3t, \quad \dots, \quad \sin t, \quad \sin 2t, \quad \sin 3t, \quad \dots$$

are of unit length and perpendicular to each other. Moreover, by the theorem in the preceding section, any element of of V can be written as a (possibly infinite) superposition of these functions. We will say that the functions

$$e_0(t) = \frac{1}{\sqrt{2}}, \quad e_1(t) = \cos t, \quad e_2(t) = \cos 2t, \quad \dots,$$

$$\hat{e}_1(t) = \sin t, \quad \hat{e}_2(t) = \sin 2t, \quad \dots$$

make up an *orthonormal basis* for V.

We saw in Section 2.3 that if $\mathbf{b}_1$, $\mathbf{b}_2$, ... , $\mathbf{b}_n$ is an orthonormal basis for R^n, then

$$\mathbf{f} \in \mathsf{R}^n \quad \Rightarrow \quad \mathbf{f} = (\mathbf{f}\cdot\mathbf{b}_1)\mathbf{b}_1 + \cdots + (\mathbf{f}\cdot\mathbf{b}_n)\mathbf{b}_n.$$

The same formula holds when R^n is replaced by V and the dot product is replaced by the inner product $\langle\cdot,\cdot\rangle$: If f is any element of V, we can write

$$f(t) = \langle f(t), e_0(t)\rangle e_0(t) + \langle f(t), e_1(t)\rangle e_1(t) + \langle f(t), e_2(t)\rangle e_2(t) + \cdots$$
$$+\langle f(t), \hat{e}_1(t)\rangle \hat{e}_1(t) + \langle f(t), \hat{e}_2(t)\rangle \hat{e}_2(t) + \cdots .$$

In other words,

$$f(t) = \langle f, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} + \langle f, \cos t\rangle \cos t + \langle f, \cos 2t\rangle \cos 2t + \ldots$$
$$+\langle f, \sin t\rangle \sin t + \langle f, \sin 2t\rangle \sin 2t + \ldots ,$$
$$= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots + b_1 \sin t + b_2 \sin 2t + \ldots ,$$

where

$$a_0 = \langle f, 1\rangle, \quad a_1 = \langle f, \cos t\rangle, \quad a_2 = \langle f, \cos 2t\rangle, \quad \ldots ,$$
$$b_1 = \langle f, \sin t\rangle, \quad b_2 = \langle f, \sin 2t\rangle, \quad \ldots .$$

Use of the inner product makes the formulae for Fourier coefficients almost impossible to forget.

We say that a function $f(t)$ is *odd* if $f(-t) = -f(t)$, *even* if $f(-t) = f(t)$. Thus

$$\sin t, \qquad \sin 2t, \qquad \sin 3t, \qquad \ldots$$

are odd functions, while

$$1, \qquad \cos t, \qquad \cos 2t, \qquad \cos 3t, \qquad \ldots$$

are even functions. Let

$$W_{\text{odd}} = \{f \in V : f \text{ is odd}\}, \qquad W_{\text{even}} = \{f \in V : f \text{ is odd}\}.$$

Then

$$f, g \in W_{\text{odd}} \qquad \Rightarrow \qquad f + g \in W_{\text{odd}} \qquad \text{and} \qquad cf \in W_{\text{odd}},$$

for every choice of real number c. Thus we can say that W_{odd} is a linear subspace of V. Similarly, W_{even} is a linear subspace of V.

It is not difficult to show that

$$f \in W_{\text{odd}}, \qquad g \in W_{\text{even}} \qquad \Rightarrow \qquad \langle f, g\rangle = 0;$$

in other words, the linear subspaces W_{odd} and W_{even} are orthogonal to each other. Indeed, under these conditions, fg is odd and hence

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt = \frac{1}{\pi}\int_{-\pi}^{0} f(t)g(t)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)g(t)dt$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-t)g(-t)(-dt) + \frac{1}{\pi}\int_{0}^{\pi} f(t)g(t)dt$$
$$= -\frac{1}{\pi}\int_{0}^{\pi} -(f(t)g(t))(-dt) + \frac{1}{\pi}\int_{0}^{\pi} f(t)g(t)dt = 0.$$

The variable of integration has been changed from t to $-t$ in the first integral of the second line.

It follows that if $f \in W_{\text{odd}}$,

$$a_0 = \langle f, 1\rangle = 0, \qquad a_n = \langle f, \cos nx\rangle = 0, \quad \text{for } n > 0.$$

Similarly, if $f \in W_{\text{even}}$,

$$b_n = \langle f, \sin nx\rangle = 0.$$

Thus for an even or odd function, half of the Fourier coefficients are automatically zero. This simple fact can often simplify the calculation of Fourier coefficients.

Exercises:

3.2.1. Evaluate the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt,$$

in the case where $f(t) = \cos t$ and $g(t) = |\sin t|$.

3.2.2. Evaluate the inner product

$$\langle f, g\rangle = \int_{1}^{e^{\pi}} \frac{1}{x} f(x)g(x)dx,$$

in the case where $f(x) = \cos(\log(x))$ and $g(x) = 1$, where log denotes the natural or base e logarithm. (Hint: Use the substitution $x = e^u$.)

3.2.3. Determine which of the following functions are even, which are odd, and which are neither even nor odd:

a. $f(t) = t^3 + 3t$.

b. $f(t) = t^2 + |t|$.

c. $f(t) = e^t$.

d. $f(t) = \frac{1}{2}(e^t + e^{-t})$.

e. $f(t) = \frac{1}{2}(e^t - e^{-t})$.

f. $f(t) = J_0(t)$, the Bessel function of the first kind.

3.3 Fourier sine and cosine series

Let $f : [0, L] \to \mathsf{R}$ be a piecewise smooth function which vanishes at 0 and L. We claim that we can express $f(t)$ as the superposition of sine functions,

$$f(t) = b_1 \sin(\pi t/L) + b_2 \sin(2\pi t/L) + \ldots + b_n \sin(n\pi t/L) + \ldots . \tag{3.10}$$

We could prove this using the theory of even and odd functions. Indeed, we can extend f to an odd function $\tilde{f} : [-L, L] \to R$ by setting

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in [0, L], \\ -f(-t), & \text{for } t \in [-L, 0], \end{cases}$$

then to a function $\hat{f} : R \to R$, which is periodic of period $2L$ by requiring that

$$\hat{f}(t + 2L) = \hat{f}(t), \qquad \text{for all } t \in \mathsf{R}.$$

The extended function lies in the linear subspace W_{odd}. It follows from the theorem in Section 3.1 that $\hat{f}$ possesses a Fourier series expansion, and from the fact that $\hat{f}$ is odd that all of the a_n's are zero. On the interval $[0, L]$, $\hat{f}$ restricts to f and the Fourier expansion of $\hat{f}$ restricts to an expansion of f of the form (3.10) which involves only sine functions. We call (3.10) the *Fourier sine series* of f.

A similar argument can be used to express a piecewise smooth function $f : [0, L] \to \mathsf{R}$ into a superposition of cosine functions,

$$f(t) = \frac{a_0}{2} + a_1 \cos(\pi t/L) + a_2 \cos(2\pi t/L) + \ldots + a_n \cos(n\pi t/L) + \ldots . \tag{3.11}$$

To obtain this expression, we first extend f to an even function $\tilde{f} : [-L, L] \to R$ by setting

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in [0, L], \\ f(-t), & \text{for } t \in [-L, 0], \end{cases}$$

then to a function $\hat{f} : R \to R$ which is periodic of period $2L$, by requiring that

$$\hat{f}(t + 2L) = \hat{f}(t), \qquad \text{for all } t \in \mathsf{R}R.$$

This time the extended function lies in the linear subspace W_{even}. It follows from the theorem in Section 3.1 that $\hat{f}$ possesses a Fourier series expansion, and from the fact that $\hat{f}$ is even that all of the b_n's are zero. On the interval $[0, L]$, $\hat{f}$ restricts to f and the Fourier expansion of $\hat{f}$ restricts to an expansion of f of the form (3.11) which involves only cosine functions. We call (3.11) the *Fourier cosine series* of f.

To generate formulae for the coefficients in the Fourier sine and cosine series of f, we begin by defining a slightly different inner product space than the one considered in the preceding section. This time, we let V be the set of piecewise smooth functions $f : [0, L] \to \mathsf{R}$ and let

$$V_0 = \{f \in V : f(0) = 0 = f(L)\},$$

a linear subspace of V. We define an inner product $\langle \cdot, \cdot \rangle$ on V by means of the formula

$$\langle f, g \rangle = \frac{2}{L} \int_0^L f(t)g(t)dt.$$

This restricts to an inner product on V_0.

Let's consider now the Fourier sine series. We have seen that any element of V_0 can be represented as a superpostion of the sine functions

$$\sin(\pi t/L), \quad \sin(2\pi t/L), \quad \ldots, \quad \sin(n\pi t/L), \quad \ldots.$$

We claim that these sine functions form an orthonormal basis for V_0 with respect to the inner product we have defined. Recall the trigonometric formulae that we used in §3.1:

$$\cos((n+m)\pi t/L) = \cos(n\pi t/L)\cos(m\pi t/L) - \sin(n\pi t/L)\sin(m\pi t/L),$$

$$\cos((n-m)\pi t/L) = \cos(n\pi t/L)\cos(m\pi t/L) + \sin(n\pi t/L)\sin(m\pi t/L).$$

Subtracting the first of these from the second and dividing by two yields

$$\sin(n\pi t/L)\sin(m\pi t/L) = \frac{1}{2}(\cos((n-m)\pi t/L) - \cos((n+m)\pi t/L),$$

and hence

$$\int_0^L \sin(n\pi t/L)\sin(m\pi t/L)dt =$$

$$\frac{1}{2}\int_0^L (\cos((n-m)\pi t/L) - \cos((n+m)\pi t/L)dt.$$

If n and m are positive integers, the integral on the right vanishes unless $n = m$, in which case the right-hand side becomes

$$\frac{1}{2}\int_0^L dt = \frac{L}{2}.$$

Hence

$$\frac{2}{L}\int_0^\pi \sin(n\pi t/L)\sin(m\pi t/L)dt = \begin{cases} 1, & \text{for } m = n, \\ 0, & \text{for } m \neq n. \end{cases} \tag{3.12}$$

Therefore, just as in the previous section, we can evaluate the coefficients of the Fourier sine series of a function $f \in V_0$ by simply projecting f onto each element of the orthonormal basis. When we do this, we find that

$$f(t) = b_1 \sin(\pi t/L) + b_2 \sin(2\pi t/L) + \ldots + \sin(n\pi t/L) + \ldots,$$

where

$$b_n = \langle f, \sin(n\pi t/L)\rangle = \frac{2}{L}\int_0^L f(t)\sin(n\pi t/L)dt. \tag{3.13}$$

We can treat the Fourier cosine series in a similar fashion. In this case, we can show that the functions

$$\frac{1}{\sqrt{2}}, \quad \cos(\pi t/L), \quad \cos(2\pi t/L), \quad \dots, \quad \cos(n\pi t/L), \quad \dots$$

form an orthonormal basis for V. Thus we can evaluate the coefficients of the Fourier cosine series of a function $f \in V$ by projecting f onto each element of this orthonormal basis. We will leave it to the reader to carry this out in detail, and simply remark that when the dust settles, one obtains the following formula for the coefficient a_n in the Fourier cosine series:

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(n\pi t/L)dt. \tag{3.14}$$

Example. First let us use (3.13) to find the Fourier sine series of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le \pi/2, \\ \pi - t, & \text{for } \pi/2 \le t \le \pi. \end{cases} \tag{3.15}$$

In this case, $L = \pi$, and according to our formula,

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} t\sin ntdt + \int_{\pi/2}^{\pi}(\pi - t)\sin ntdt\right].$$

We use integration by parts to obtain

$$\int_0^{\pi/2} t\sin ntdt = \left[\frac{-t}{n}\cos nt\right]\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{n}\cos ntdt$$

$$= \frac{-\pi\cos(n\pi/2)}{2n} + \frac{1}{n^2}[\sin nt]|_0^{\pi/2} = \frac{-\pi\cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2},$$

while

$$\int_{\pi/2}^{\pi}(\pi - t)\sin ntdt = \left[\frac{-(\pi - t)}{n}\cos(nt)\right]\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\frac{1}{n}\cos ntdt$$

$$= \frac{\pi\cos(n\pi/2)}{2n} - \frac{1}{n^2}[\sin nt]|_{\pi/2}^{\pi} = \frac{\pi\cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2}.$$

Thus

$$b_n = \frac{4\sin(n\pi/2)}{\pi n^2},$$

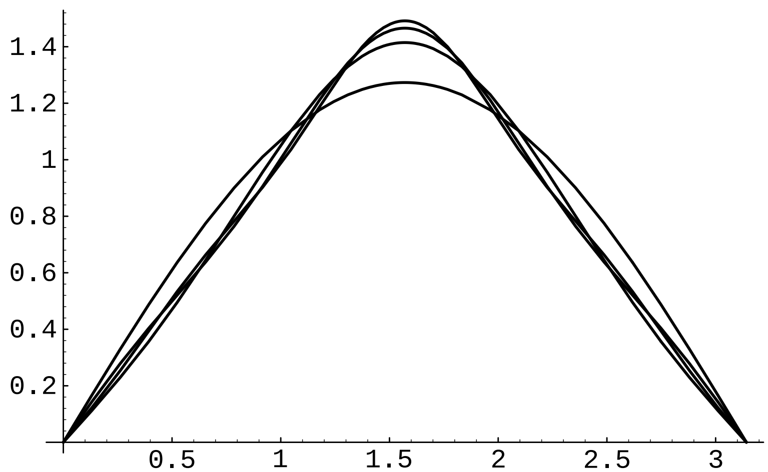

Figure 3.3: A graph of the Fourier sine approximations ϕ_1, ϕ_3, ϕ_5 and ϕ_7.

and the Fourier sine series of $f(t)$ is

$$f(t) = \frac{4}{\pi}\sin t - \frac{4}{9\pi}\sin 3t + \frac{4}{25\pi}\sin 5t - \frac{4}{49\pi}\sin 7t + \dots . \tag{3.16}$$

The trigonometric polynomials

$$\phi_1(t) = \frac{4}{\pi}\sin t, \quad \phi_3(t) = \frac{4}{\pi}\sin t - \frac{4}{9\pi}\sin 3t, \dots$$

are better and better approximations to the function $f(t)$.

To find the Fourier cosine series of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le \pi/2, \\ \pi - t, & \text{for } \pi/2 \le t \le \pi, \end{cases}$$

we first note that

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} t dt + \int_{\pi/2}^{\pi} (\pi - t) dt\right] = \frac{2}{\pi}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \frac{1}{2}\left(\frac{\pi}{2}\right)^2\right] = \frac{\pi}{2}.$$

To find the other a_n's, we use (3.14):

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} t\cos nt dt + \int_{\pi/2}^{\pi} (\pi - t)\cos nt dt\right].$$

This time, integration by parts yields

$$\int_0^{\pi/2} t\cos nt dt = \left[\frac{t}{n}\sin nt\right]\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{n}\sin nt dt$$

$$= \frac{\pi \sin(n\pi/2)}{2n} + \frac{1}{n^2}[\cos nt]|_0^{\pi/2}$$

$$= \frac{\pi \sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2}$$

while

$$\int_{\pi/2}^{\pi} (\pi - t)\cos ntdt = \left[\frac{(\pi - t)}{n}\sin(nt)\right]\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{1}{n}\sin ntdt$$

$$= \frac{-\pi \sin(n\pi/2)}{2n} - \frac{1}{n^2}[\cos nt]|_{\pi/2}^{\pi}$$

$$= \frac{-\pi \sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2}\cos(n\pi).$$

Thus when $n \geq 1$,

$$a_n = \frac{2}{\pi n^2}[2\cos(n\pi/2) - 1 - 1(-1)^n],$$

and the Fourier sine series of $f(t)$ is

$$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\cos 2t - \frac{2}{9\pi}\cos 6t - \frac{2}{25\pi}\cos 10t - \dots. \tag{3.17}$$

Note that we have expanded exactly the same function $f(t)$ on the interval $[0, \pi]$ as either a superposition of sines in (3.16) or as a superposition of cosines in (3.17).

Exercises:

3.3.1.a. Find the Fourier sine series of the following function defined on the interval $[0, \pi]$:

$$f(t) = \begin{cases} 4t, & \text{for } 0 \leq t < \pi/2, \\ 4\pi - 4t, & \text{for } \pi/2 \leq t \leq \pi. \end{cases}$$

b. Find the Fourier cosine series of the same function.

3.3.2.a. Find the Fourier sine series of the following function defined on the interval $[0, \pi]$:

$$f(t) = t(\pi - t).$$

b. Find the Fourier cosine series of the same function.

3.3.3. Find the Fourier sine series of the following function defined on the interval $[0, 10]$:

$$f(t) = \begin{cases} t, & \text{for } 0 \leq t < 5, \\ 10 - t, & \text{for } 5 \leq t \leq 10. \end{cases}$$

3.3.4. Find the Fourier sine series of the following function defined on the interval $[0, 1]$:

$$f(t) = 5t(1 - t).$$

3.3.5.(For students with access to Mathematica) Find numerical approximations to the first ten coefficients of the Fourier sine series for the function

$$f(t) = t + t^2 - 2t^3,$$

defined for t in the interval $[0, 1]$, by running the following Mathematica program

```
f[n_] := 2 NIntegrate[(t + t^2 - 2 t^3) Sin[n Pi t], {t,0,1}];
b = Table[f[n], {n,1,10}]
```

3.4 Complex version of Fourier series*

We have seen that if $f : R \to R$ is a well-behaved function which is periodic of period 2π, f can be expanded in a Fourier series

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos(2t) + \dots$$
$$+ b_1 \sin t + b_2 \sin(2t) + \dots .$$

We say that this is the *real form* of the Fourier series. It is often convenient to recast this Fourier series in *complex form* by means of the Euler formula, which states that

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

It follows from this formula that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta, \qquad e^{-i\theta} + e^{-i\theta} = 2i\sin\theta,$$

or

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}.$$

Hence the Fourier expansion of f can be rewritten as

$$f(t) = \frac{a_0}{2} + \frac{a_1}{2}(e^{it} + e^{-it}) + \frac{a_2}{2}(e^{2it} + e^{-2it}) + \dots$$
$$+ \frac{b_1}{2i}(e^{it} - e^{-it}) + \frac{b_2}{2i}(e^{2it} - e^{-2it}) + \dots ,$$

or

$$f(t) = \dots + c_{-2}e^{-2it} + c_{-1}e^{-it} + c_0 + c_1 e^{it} + c_2 e^{2it} + \dots , \tag{3.18}$$

where the c_k's are the complex numbers defined by the formulae

$$c_0 = \frac{a_0}{2}, \quad c_1 = \frac{a_1 - ib_1}{2}, \quad c_2 = \frac{a_2 - ib_2}{2}, \quad \dots ,$$

$$c_{-1} = \frac{a_1 + ib_1}{2}, \quad c_{-2} = \frac{a_2 + ib_2}{2}, \quad \ldots .$$

If $k \neq 0$, we can solve for a_k and b_k:

$$a_k = c_k + c_{-k}, \qquad b_k = i(c_k - c_{-k}). \tag{3.19}$$

It is not difficult to check the following integral formula via direct integration:

$$\int_{-\pi}^{\pi} e^{imt} e^{-int} dt = \begin{cases} 2\pi, & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases} \tag{3.20}$$

If we multiply both sides of (3.18) by e^{-ikt}, integrate from $-\pi$ to π and apply (3.20), we obtain

$$\int_{-\pi}^{\pi} f(t) e^{-ikt} dt = 2\pi c_k,$$

which yields the formula for coefficients of the complex form of the Fourier series:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt. \tag{3.21}$$

Example. Let us use formula (3.21) to find the complex Fourier coefficients of the function

$$f(t) = t \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π. Then

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} t e^{-ikt} dt.$$

We apply integration by parts with $u = t$, $dv = e^{-ikt}dt$, $du = dt$ and $v = (i/k)e^{-ikt}$:

$$c_k = \frac{1}{2\pi} \left[(it/k) e^{-ikt} |_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (i/k) e^{-ikt} dt \right]$$
$$= \frac{1}{2\pi} \left[\frac{\pi i}{k} e^{-i\pi k} + \frac{\pi i}{k} e^{i\pi k} \right] = i \frac{(-1)^k}{k}.$$

It follows from (3.19) that

$$a_k = (c_k + c_{-k}) = 0, \qquad b_k = i(c_k - c_{-k}) = -2\frac{(-1)^k}{k}.$$

It is often the case that the complex form of the Fourier series is far simpler to calculate than the real form. One can then use (3.19) to find the real form of the Fourier series.

Exercise:

3.4.1. Prove the integral formula (3.20), presented in the text.

3.4.2. a. Find the complex Fourier coefficients of the function

$$f(t) = t^2 \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π.

b. Use (3.19) to find the real form of the Fourier series.

3.4.3. a. Find the complex Fourier coefficients of the function

$$f(t) = t(\pi - t) \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π.

b. Use (3.19) to find the real form of the Fourier series.

3.4.4. Show that if a function $f : R \to R$ is smooth and periodic of period $2\pi L$, we can write the Fourier expansion of f as

$$f(t) = \ldots + c_{-2}e^{-2it/L} + c_{-1}e^{-it/L} + c_0 + c_1 e^{it/L} + c_2 e^{2it/L} + \ldots,$$

where

$$c_k = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} f(t)e^{-ikt/L}dt.$$

3.5 Fourier transforms*

One of the problems with the theory of Fourier series presented in the previous sections is that it applies only to periodic functions. There are many times when one would like to divide a function which is not periodic into a superposition of sines and cosines. The Fourier transform is the tool often used for this purpose.

The idea behind the Fourier transform is to think of $f(t)$ as vanishing outside a very long interval $[-\pi L, \pi L]$. The function can be extended to a periodic function $f(t)$ such that $f(t + 2\pi L) = f(t)$. According to the theory of Fourier series in complex form (see Exercise 3.4.4),

$$f(t) = \ldots + c_{-2}e^{-2it/L} + c_{-1}e^{-it/L} + c_0 + c_1 e^{it/L} + c_2 e^{2it/L} + \ldots,$$

where the c_k's are the complex numbers.

Definition. If $f : \mathsf{R} \to \mathsf{R}$ is a piecewise continuous function which vanishes outside some finite interval, its *Fourier transform* is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt. \tag{3.22}$$

The integral in this formula is said to be *improper* because it is taken from $-\infty$ to ∞; it is best to regard it as a limit,

$$\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt = \lim_{L\to\infty}\int_{-\pi L}^{\pi L} f(t)e^{-i\xi t}dt.$$

To explain (3.22), we suppose that f vanishes outside the interval $[-\pi L, \pi L]$. We can extend the restriction of f to this interval to a function which is periodic of period $2\pi L$. Then

$$\hat{f}(k/L) = \int_{-\infty}^{\infty} f(t) e^{-ikt/L} dt = \int_{-\pi L}^{\pi L} \tilde{f}(t) e^{-ikt/L} dt$$

represents $2\pi L$ times the Fourier coefficient of this extension of frequency k/L; indeed, it follows from (3.21) that we can write

$$f(t) = \ldots + c_{-2} e^{-2it/L} + c_{-1} e^{-it/L} + c_0 + c_1 e^{it/L} + c_2 e^{2it/L} + \ldots ,$$

for $t \in [-\pi L, \pi L]$, where

$$c_k = \frac{1}{2\pi L} \hat{f}(k/L),$$

or alternatively,

$$f(t) = \ldots + \frac{1}{2\pi L} \hat{f}(-2/L) e^{-2it/L} + \frac{1}{2\pi L} \hat{f}(-1/L) e^{-it/L} +$$
$$+ \frac{1}{2\pi L} \hat{f}(0) + \frac{1}{2\pi L} \hat{f}(1/T) e^{it/L} + \frac{1}{2\pi L} \hat{f}(2/L) e^{2it/L} + \ldots .$$

In the limit as $L \to \infty$, it can be shown that this last sum approaches an improper integral, and our formula becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi t} d\xi. \tag{3.23}$$

Equation (3.23) is called the *Fourier inversion formula*. If we make use of Euler's formula, we can write the Fourier inversion formula in terms of sines and cosines,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \cos \xi t d\xi + \frac{i}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \sin \xi t d\xi,$$

a superposition of sines and cosines of various frequencies.

Equations (3.22) and (3.22) allow one to pass back and forth between a given function and its representation as a superposition of oscillations of various frequencies. Like the Laplace transform, the Fourier transform is often an effective tool in finding explicit solutions to differential equations.

Exercise:

3.5.1. Find the Fourier transform of the function $f(t)$ defined by

$$f(t) = \begin{cases} 1, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 4

Partial Differential Equations

4.1 Overview

A *partial differential equation* is an equation which contains partial derivatives, such as the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

in which u is regarded as a function of x and t. Unlike the theory of ordinary differential equations which centers upon one key theorem—the fundamental existence and uniqueness theorem—there is no real unified theory of partial differential equations. Instead, each type of partial differential equations exhibits its own special features, which usually mirror the physical phenomena which the equation was first used to model.

Many of the foundational theories of physics and engineering are expressed by means of systems of partial differential equations. The reader may have heard some of these equations mentioned in previous courses in physics. Fluid mechanics is often formulated by the Euler equations of motion or the so-called Navier-Stokes equations, electricity and magnetism by Maxwell's equations, general relativity by Einstein's field equations. It is therefore important to develop techniques that can be used to solve a wide variety of partial differential equations.

In this chapter, we will give two important simple examples of partial differential equations, the heat equation and the wave equation, and we will show how to solve them by the techniques of "separation of variables" and Fourier analysis. Higher dimensional examples will be given in the following chapter. We will see that just as in the case of ordinary differential equations, there is an important dichotomy between linear and nonlinear equations. The techniques of separation of variables and Fourier analysis are effective only for linear partial differential equations. Nonlinear partial differential equations are far more difficult to solve, and form a key topic of contemporary mathematical research.[1]

Our first example is the equation governing propagation of heat in a bar of length L. We imagine that the bar is located along the x-axis and we let

$$u(x,t) = \text{temperature of the bar at the point } x \text{ at time } t.$$

Heat in a small segment of a homogeneous bar is proportional to temperature, the constant of proportionality being determined by the density and specific heat of the material making up the bar. More generally, if $\sigma(x)$ denotes the specific heat at the point x and $\rho(x)$ is the density of the bar at x, then the heat within the region D_{x_1,x_2} between x_1 and x_2 is given by the formula

$$\text{Heat within } D_{x_1,x_2} = \int_{x_1}^{x_2} \rho(x)\sigma(x)u(x,t)dx.$$

To calculate the rate of change of heat within D_{x_1,x_2} with respect to time, we simply differentiate under the integral sign:

$$\frac{d}{dt}\left[\int_{x_1}^{x_2} \rho(x)\sigma(x)u(x,t)dx\right] = \int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx.$$

Now heat is a form of energy, and conservation of energy implies that if no heat is being created or destroyed in D_{x_1,x_2}, the rate of change of heat within D_{x_1,x_2} is simply the rate at which heat enters D_{x_1,x_2}. Hence the rate at which heat leaves D_{x_1,x_2} is given by the exp

$$\text{Rate at which heat leaves } D_{x_1,x_2} = -\int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx. \tag{4.1}$$

(More generally, if heat is being created within D_{x_1,x_2}, say by a chemical reaction, at the rate $\mu(x)u(x,t) + \nu(x)$ per unit volume, then the rate at which heat leaves D_{x_1,x_2} is

$$-\int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx + \int_{x_1}^{x_2} (\mu(x)u(x,t) + \nu(x))dx.)$$

On the other hand, the rate of heat flow $F(x,t)$ is proportional to the partial derivative of temperature,

$$F(x,t) = -\kappa(x)\frac{\partial u}{\partial x}(x,t), \tag{4.2}$$

where $\kappa(x)$ is the thermal conductivity of the bar at x. Thus we find that the rate at which heat leaves the region D_{x_1,x_2} is also given by the formula

$$F(x_2,t) - F(x_1,t) = \int_{x_1}^{x_2} \frac{\partial F}{\partial x}(x,t)dx. \tag{4.3}$$

[1]Further reading can be found in the many excellent upper-division texts on partial differential equations. We especially recommend Mark Pinsky, *Partial differential equations and boundary-value problems with applications*, 2nd edition, McGraw-Hill, 1991. An excellent but much more advanced book is Michael Taylor, *Partial differential equations: basic theory*, Springer, New York, 1996.

Comparing the two formulae (4.1) and (4.3), we find that

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial x}(x,t)dx = -\int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx.$$

This equation is true for *all* choices of x_1 and x_2, so the integrands on the two sides must be equal:

$$\frac{\partial F}{\partial x} = -\rho(x)\sigma(x)\frac{\partial u}{\partial t}.$$

It follows from (4.2) that

$$\frac{\partial}{\partial x}\left(-\kappa(x)\frac{\partial u}{\partial x}\right) = -\rho(x)\sigma(x)\frac{\partial u}{\partial t}.$$

In this way, we obtain the *heat equation*

$$\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right). \tag{4.4}$$

In the more general case in which heat is being created at the rate $\mu(x)u(x,t) + \nu(x)$ per unit length, one could show that heat flow is modeled by the equation

$$\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right) + \mu(x)u + \nu(x). \tag{4.5}$$

In the special case where the bar is *homogeneous*, i.e. its properties are the same at every point, $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are constants, say σ and κ respectively, and (4.4) becomes

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho\sigma}\frac{\partial^2 u}{\partial x^2}. \tag{4.6}$$

This is our simplest example of a linear partial differential equation. Although its most basic application concerns diffusion of heat, it arises in many other contexts as well. For example, a slight modification of the heat equation was used by Black and Scholes to price derivatives in financial markets.[2]

Exercises:

4.1.1. Study of heat flow often leads to "boundary-value problems" for ordinary differential equations. Indeed, in the "steady-state" case, in which u is independent of time, equation (4.5) becomes

$$\frac{d}{dx}\left(\kappa(x)\frac{du}{dx}(x)\right) + \mu(x)u(x) + \nu(x) = 0,$$

[2] A description of the Black-Scholes technique for pricing puts and calls is given in Paul Wilmott, Sam Howison and Jeff Dewynne, *The mathematics of financial derivatives*, Cambridge Univ. Press, 1995.

a linear *ordinary* differential equation with variable coefficients. Suppose now that the temperature is specified at the two endpoints of the bar, say

$$u(0) = \alpha, \quad u(L) = \beta.$$

Our physical intuition suggests that the steady-state heat equation should have a unique solution with these boundary conditions.

a. Solve the following special case of this boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} = 0, \quad u(0) = 70, \quad u(1) = 50.$$

b. Solve the following special case of this boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} - u = 0, \quad u(0) = 70, \quad u(1) = 50.$$

c. Solve the following special case of this boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} + x(1-x) = 0, \quad u(0) = 0, \quad u(1) = 0.$$

d. (For students with access to Mathematica) Use Mathematica to graph the solution to the following boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} + (1+x^2)u = 0, \quad u(0) = 50, \quad u(1) = 100.$$

You can do this by running the Mathematica program:

```
a = 0; b = 1; alpha = 50; beta = 100;
sol = NDSolve[ {u''[x] + (1 + x∧2) u[x] == 0,
u[a] == alpha, u[b] == beta.}, u, {x,a,b}];
Plot[ Evaluate[ u[x] /.  sol], {x,a,b}]
```

4.1.2.a. Show that the function

$$u_0(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

is a solution to the heat equation (4.6) for $t > 0$ in the case where $\kappa/(\rho\sigma) = 1$.

b. Use the chain rule with intermediate variables $\bar{x} = x - a$, $\bar{t} = t$ to show that

$$u_a(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-a)^2/4t}$$

is also a solution to the heat equation.

c. Show that

$$\int_{-\infty}^{\infty} u_0(x,t)dx = 1.$$

Hint: Let I denote the integral on the left hand side and note that

$$I^2 = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/4t}dxdy.$$

Then transform this last integral to polar coordinates.

d. Use Mathematica to sketch $u_0(x,t)$ for various values of t. What can you say about the behaviour of the function $u_0(x,t)$ as $t \to 0$?

e. By differentiating under the integral sign show that if $h : R \to R$ is any smooth function which vanishes outside a finite interval $[-L, L]$, then

$$u(x,t) = \int_{-\infty}^{\infty} u_a(x,t)h(a)da \tag{4.7}$$

is a solution to the heat equation.

REMARK: In more advanced courses it is shown that (4.7) approaches $h(x)$ as $t \to 0$. In fact, (4.7) gives a formula (for values of t which are greater than zero) for the unique solution to the heat equation on the infinite line which satisfies the initial condition $u(x,0) = h(x)$. In the next section, we will see how to solve the initial value problem for a rod of finite length.

4.2 The initial value problem for the heat equation

We will now describe how to use the Fourier sine series to find the solution to an *initial value problem* for the heat equation in a rod of length L which is insulated along the sides, whose ends are kept at zero temperature. We expect that there should exist a unique function $u(x,t)$, defined for $0 \leq x \leq L$ and $t \geq 0$ such that

1. $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{4.8}$$

   where c is a constant.

2. $u(x,t)$ satisfies the boundary condition $u(0,t) = u(L,t) = 0$, in other words, the temperature is zero at the endpoints. (This is sometimes called the *Dirichlet boundary condition.*)

3. $u(x,t)$ satisfies the initial condition $u(x,0) = h(x)$, where $h(x)$ is a given function, the initial temperature of the rod.

In more advanced courses, it is proven that this initial value problem does in fact have a unique solution. We will shortly see how to find that solution.

Note that the heat equation itself and the boundary condition are *homogeneous* and *linear*—this means that if u_1 and u_2 satisfy these conditions, so does $c_1u_1 + c_2u_2$, for any choice of constants c_1 and c_2. Thus homogeneous linear conditions satisfy the principal of superposition.

Our method makes use of the dichotomy into homogeneous and nonhomogeneous conditions:

Step I. We find all of the solutions to the homogeneous linear conditions of the special form

$$u(x,t) = f(x)g(t).$$

By the superposition principle, an arbitrary linear superposition of these solutions will still be a solution.

Step II. We find the particular solution which satisfies the nonhomogeneous condition by Fourier analysis.

Let us first carry out Step I. We substitute $u(x,t) = f(x)g(t)$ into the heat equation (4.8) and obtain

$$f(x)g'(t) = c^2 f''(x)g(t).$$

Now we separate variables, putting all the functions involving t on the left, all the functions involving x on the right:

$$\frac{g'(t)}{g(t)} = c^2 \frac{f''(x)}{f(x)}.$$

The left-hand side of this equation does not depend on x, while the right-hand side does not depend on t. Hence neither side can depend upon either x or t. In other words, the two sides must equal a constant, which we denote by λ and call the *separating constant*. Our equation now becomes

$$\frac{g'(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda,$$

which separates into two ordinary differential equations,

$$\frac{g'(t)}{c^2 g(t)} = \lambda, \quad \text{or} \quad g'(t) = \lambda c^2 g(t), \tag{4.9}$$

and

$$\frac{f''(x)}{f(x)} = \lambda, \quad \text{or} \quad f''(x) = \lambda f(x). \tag{4.10}$$

The homogeneous boundary condition $u(0,t) = u(L,t) = 0$ becomes

$$f(0)g(t) = f(L)g(t) = 0.$$

If $g(t)$ is not identically zero,

$$f(0) = f(L) = 0.$$

(If $g(t)$ is identically zero, then so is $u(x,t)$, and we obtain only the trivial solution $u \equiv 0$.)

Thus to find the nontrivial solutions to the homogeneous linear part of the problem requires us to find the nontrivial solutions to a boundary value problem for an ordinary differential equation:

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), \quad f(0) = 0 = f(L). \tag{4.11}$$

We will call (4.11) the *eigenvalue* problem for the differential operator

$$\mathbf{L} = \frac{d^2}{dx^2}$$

acting on the space V_0 of well-behaved functions $f : [0,L] \to \mathsf{R}$ which vanish at the endpoints 0 and L.

We need to consider three cases. (As it turns out, only one of these will actually yield nontrivial solutions to our eigenvalue problem.)

Case 1: $\lambda = 0$. In this case, the eigenvalue problem (4.11) becomes

$$f''(x) = 0, \quad f(0) = 0 = f(L).$$

The general solution to the differential equation is $f(x) = ax + b$, and the only particular solution which satisfies the boundary condition is $f = 0$, the trivial solution.

Case 2: $\lambda > 0$. In this case, the differential equation

$$f''(x) = \lambda f(x), \quad \text{or} \quad f''(x) - \lambda f(x) = 0$$

has the general solution

$$f(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

It is convenient for us to change basis in the linear space of solutions, using

$$\cosh(\sqrt{\lambda}x) = \frac{1}{2}(e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x}), \qquad \sinh(\sqrt{\lambda}x) = \frac{1}{2}(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x})$$

instead of the exponentials. Then we can write

$$f(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x),$$

with new constants of integration a and b. We impose the boundary conditions: first

$$f(0) = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad f(x) = b \sinh(\sqrt{\lambda} x),$$

and then

$$f(L) = 0 \quad \Rightarrow \quad b \sinh(\sqrt{\lambda} L) = 0 \quad \Rightarrow \quad b = 0 \quad \Rightarrow \quad f(x) = 0,$$

so we obtain no nontrivial solutions in this case.

Case 3: $\lambda < 0$. In this case, we set $\omega = \sqrt{-\lambda}$, and rewrite the eigenvalue problem as

$$f''(x) + \omega^2 f(x) = 0, \quad f(0) = 0 = f(L).$$

We recognize here our old friend, the differential equation of simple harmonic motion. We remember that the differential equation has the general solution

$$f(x) = a \cos(\omega x) + b \sin(\omega x).$$

Once again

$$f(0) = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad f(x) = b \sin(\omega x).$$

Now, however,

$$f(L) = 0 \quad \Rightarrow \quad b \sin(\omega L) \quad \Rightarrow \quad b = 0 \text{ or } \sin(\omega L) = 0,$$

and hence either $b = 0$ and we obtain only the trivial solution or $\sin(\omega L) = 0$. The latter possibility will occur if $\omega L = n\pi$, or $\omega = (n\pi/L)$, where n is an integer. In this case, we obtain

$$f(x) = b \sin(n\pi x/L).$$

Therefore, we conclude that the only nontrivial solutions to (4.11) are constant multiples of

$$f(x) = \sin(n\pi x/L), \quad \text{with} \quad \lambda = -(n\pi/L)^2, \quad n = 1, 2, 3, \ldots .$$

For each of these solutions, we need to find a corresponding $g(t)$ solving equation (4.9),

$$g'(t) = \lambda c^2 g(t),$$

where $\lambda = -(n\pi/L)^2$. This is just the equation of exponential decay, and has the general solution

$$g(t) = b e^{-(nc\pi/L)^2 t},$$

where a is a constant of integration. Thus we find that the nontrivial product solutions to the heat equation together with the homogeneous boundary condition $u(0, t) = 0 = u(L, t)$ are constant multiples of

$$u_n(x, t) = \sin(n\pi x/L) e^{-(nc\pi/L)^2 t}.$$

It follows from the principal of superposition that

$$u(x,t) = b_1 \sin(\pi x/L) e^{-(c\pi/L)^2 t} + b_2 \sin(2\pi x/L) e^{-(2c\pi/L)^2 t} + \dots \tag{4.12}$$

is a solution to the heat equation together with its homogeneous boundary conditions, for arbitrary choice of the constants b_1, b_2, ...

Step II consists of determining the constants b_n in (4.12) so that the initial condition $u(x,0) = h(x)$ is satisfied. Setting $t = 0$ in (4.12) yields

$$h(x) = u(x,0) = b_1 \sin(\pi x/L) + b_2 \sin(2\pi x/L) + \dots .$$

It follows from the theory of the Fourier sine series that h can indeed be represented as a superposition of sine functions, and we can determine the b_n's as the coefficients in the Fourier sine series of h. Using the techniques described in Section 3.3, we find that

$$b_n = \frac{2}{L} \int_0^L h(x) \sin(n\pi x/L) dx.$$

Example 1. Suppose that we want to find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the initial-value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = h(x) = 4\sin x + 2\sin 2x + 7\sin 3x.$$

In this case, the nonvanishing coefficients for the Fourier sine series of h are

$$b_1 = 4, \quad b_2 = 2, \quad b_3 = 7,$$

so the solution must be

$$u(x,t) = 4\sin x e^{-t} + 2\sin 2x e^{-4t} + 7\sin 3x e^{-9t}.$$

Example 2. Suppose that we want to find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the initial-value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = h(x),$$

where

$$h(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

We saw in Section 3.3 that the Fourier sine series of h is

$$h(x) = \frac{4}{\pi}\sin x - \frac{4}{9\pi}\sin 3x + \frac{4}{25\pi}\sin 5x - \frac{4}{49\pi}\sin 7x + \dots ,$$

and hence

$$u(x,t) = \frac{4}{\pi}\sin x e^{-t} - \frac{4}{9\pi}\sin 3x e^{-9t} + \frac{4}{25\pi}\sin 5x e^{-25t} - \dots .$$

Exercises:

4.2.1. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin 2x.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f''(x) = \lambda f(x), \quad f(0) = 0 = f(\pi)$$

are

$$\lambda = -n^2, \qquad f(x) = b_n \sin nx, \qquad \text{for } n = 1, 2, 3, \ldots,$$

where the b_n's are constants.

4.2.2. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x - 5\sin 3x.$$

4.2.3. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x(\pi - x).$$

(In this problem you need to find the Fourier sine series of $h(x) = x(\pi - x)$.)

4.2.4. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x.$$

4.2.5. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$(t+1)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x.$$

4.2.6.a. Find the function $w(x)$, defined for $0 \leq x \leq \pi$, such that

$$\frac{d^2 w}{dx^2} = 0, \qquad w(0) = 10, \qquad w(\pi) = 50.$$

b. Find the general solution to the following boundary value problem for the heat equation: Find the functions $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 10, \qquad u(\pi,t) = 50. \tag{4.13}$$

(Hint: Let $v = u - w$, where w is the solution to part a. Determine what conditions v must satisfy.)

c. Find the particular solution to (4.13) which in addition satisfies the initial condition

$$u(x,0) = 10 + \frac{40}{\pi}x + 2\sin x - 5\sin 2x.$$

4.2.7.a. Find the eigenvalues and corresponding eigenfunctions for the differential operator

$$\mathbf{L} = \frac{d^2}{dt^2} + 3,$$

which acts on the space V_0 of well-behaved functions $f : [0,\pi] \to \mathsf{R}$ which vanish at the endpoints 0 and π by

$$\mathbf{L}(f) = \frac{d^2 f}{dt^2} + 3f.$$

b. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 3u, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x.$$

4.2.8. The method described in this section can also be used to solve an initial value problem for the heat equation in which the Dirichlet boundary condition $u(0,t) = u(L,t) = 0$ is replaced by the *Neumann boundary condition*

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0. \tag{4.14}$$

(Physically, this corresponds to insulated endpoints, from which no heat can enter or escape.) In this case separation of variables leads to a slightly different eigenvalue problem, which consists of finding the nontrivial solutions to

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), \quad f'(0) = 0 = f'(L).$$

a. Solve this eigenvalue problem. (Hint: The solution should involve cosines instead of sines.)

b. Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the Neumann boundary condition (4.14).

c. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, such that:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = 3\cos x + 7\cos 2x.$$

4.2.9. We can also treat a mixture of Dirichlet and Neumann conditions, say

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0. \tag{4.15}$$

In this case separation of variables leads to the eigenvalue problem which consists of finding the nontrivial solutions to

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), \quad f(0) = 0 = f'(L).$$

a. Solve this eigenvalue problem.

b. Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the mixed boundary condition (4.15).

c. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, such that:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = 4\sin(x/2) + 12\sin(3x/2).$$

4.3 Numerical solutions to the heat equation

There is another method which is sometimes used to treat the initial value problem described in the preceding section, a numerical method based upon "finite differences." Although it yields only approximate solutions, it can be applied in some cases with variable coefficients when it would be impossible to apply Fourier analysis in terms of sines and cosines. However, for simplicity, we will describe only the case where ρ and k are constant, and in fact we will assume that $c^2 = L = 1$.

Thus we seek the function $u(x,t)$, defined for $0 \leq x \leq 1$ and $t \geq 0$, which solves the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = h(x)$, where $h(x)$ is a given function, representing the initial temperature.

For any fixed choice of t_0 the function $u(x,t_0)$ is an element of V_0, the space of piecewise smooth functions defined for $0 \leq x \leq 1$ which vanish at the endpoints.

Our idea is to replace the "infinite-dimensional" space V_0 by a finite-dimensional Euclidean space R^{n-1} and reduce the partial differential equation to a system of ordinary differential equations. This corresponds to utilizing a discrete model for heat flow rather than a continuous one.

For $0 \leq i \leq n$, let $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \text{the temperature at } x_i \text{ at time } t.$$

Since $u_0(t) = 0 = u_n(t)$ by the boundary conditions, the temperature at time t is specified by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix},$$

a vector-valued function of one variable. The initial condition becomes

$$\mathbf{u}(0) = \mathbf{h}, \qquad \text{where} \qquad \mathbf{h} = \begin{pmatrix} h(x_1) \\ h(x_2) \\ \cdot \\ h(x_{n-1}) \end{pmatrix}.$$

We can approximate the first-order partial derivative by a difference quotient:

$$\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) \doteq \frac{u_{i+1}(t) - u_i(t)}{x_{i+1} - x_i} = \frac{[u_{i+1}(t) - u_i(t)]}{1/n} = n[u_{i+1}(t) - u_i(t)].$$

Similarly, we can approximate the second-order partial derivative:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_i, t) &\doteq \frac{\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) - \frac{\partial u}{\partial x}\left(\frac{x_{i-1} + x_i}{2}, t\right)}{1/n} \\ &\doteq n\left[\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) - \frac{\partial u}{\partial x}\left(\frac{x_{i-1} + x_i}{2}, t\right)\right] \\ &\doteq n^2[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)]. \end{aligned}$$

Thus the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

can be approximated by a system of ordinary differential equations

$$\frac{du_i}{dt} = n^2(u_{i-1} - 2u_i + u_{i+1}).$$

This is a first order linear system which can be presented in vector form as

$$\frac{d\mathbf{u}}{dt} = n^2 P\mathbf{u}, \quad \text{where} \quad P = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -2 \end{pmatrix},$$

the last matrix having $n-1$ rows and $n-1$ columns. Finally, we can rewrite this as

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \qquad \text{where} \qquad A = n^2 P, \tag{4.16}$$

a system exactly like the ones studied in Section 2.5. In the limit as $n \to \infty$ one can use the Mathematica program of §2.6 to check that the eigenvalues of A approach the eigenvalues of d^2/dx^2 as determined in the preceding section, and the eigenvectors approximate more and more closely the standard orthonormal basis of sine functions.

One could continue constructing a numerical method for solution of our initial value problem by means of another discretization, this time in the time direction. We could do this via the familiar Cauchy-Euler method for finding numerical solutions to the linear system (4.16). This method for finding approximate solutions to the heat equation is often called the *method of finite differences*. With sufficient effort, one could construct a computer program, using Mathematica or some other software package, to implement it.

More advanced courses on numerical analysis often treat the finite difference method in detail.[3] For us, however, the main point of the method of finite differences is that it provides considerable insight into the theory behind the heat equation. It shows that the heat equation can be thought of as arising from a system of ordinary differential equations when the number of dependent variables goes to infinity. It is sometimes the case that either a partial differential equation or a system of ordinary differential equations with a large or even infinite number of unknowns can give an effective model for the same physical phenomenon. This partially explains, for example, why quantum mechanics possesses two superficially different formulations, via Schrödinger's partial differential equation or via "infinite matrices" in Heisenberg's "matrix mechanics."

4.4 The vibrating string

Our next goal is to derive the equation which governs the motion of a vibrating string. We consider a string of length L stretched out along the x-axis, one end of the string being at $x = 0$ and the other being at $x = L$. We assume that the string is free to move only in the vertical direction. Let

$$u(x,t) = \text{vertical displacement of the string at the point } x \text{ at time } t.$$

We will derive a partial differential equation for $u(x,t)$. Note that since the ends of the string are fixed, we must have $u(0,t) = 0 = u(L,t)$ for all t.

It will be convenient to use the "configuration space" V_0 described in Section 3.3. An element $u(x) \in V_0$ represents a configuration of the string at some

[3] For further discussion of this method one can refer to numerical analysis books, such as Burden and Faires, *Numerical analysis*, Seventh edition, Brooks Cole Publishing Company, 2000.

instant of time. We will assume that the *potential energy* in the string when it is in the configuration $u(x)$ is

$$V(u(x)) = \int_0^L \frac{T}{2}\left(\frac{du}{dx}\right)^2 dx, \tag{4.17}$$

where T is a constant, called the *tension* of the string.

Indeed, we could imagine that we have devised an experiment that measures the potential energy in the string in various configurations, and has determined that (4.17) does indeed represent the total potential energy in the string. On the other hand, this expression for potential energy is quite plausible for the following reason: We could imagine first that the amount of energy in the string should be proportional to the amount of stretching of the string, or in other words, proportional to the length of the string. From vector calculus, we know that the length of the curve $u = u(x)$ is given by the formula

$$\text{Length} = \int_0^L \sqrt{1 + (du/dx)^2}dx.$$

But when du/dx is small,

$$\left[1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2\right]^2 = 1 + \left(\frac{du}{dx}\right)^2 + \text{a small error},$$

and hence

$$\sqrt{1 + (du/dx)^2} \quad \text{is closely approximated by} \quad 1 + \frac{1}{2}(du/dx)^2.$$

Thus to a first order of approximation, the amount of energy in the string should be proportional to

$$\int_0^L \left[1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2\right] dx = \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + \text{constant}.$$

Letting T denote the constant of proportionality yields

$$\text{energy in string} = \int_0^L \frac{T}{2}\left(\frac{du}{dx}\right)^2 dx + \text{constant}.$$

Potential energy is only defined up to addition of a constant, so we can drop the constant term to obtain (4.17).

The force acting on a portion of the string when it is in the configuration $u(x)$ is determined by an element $F(x)$ of V_0. We imagine that the force acting on the portion of the string from x to $x + dx$ is $F(x)dx$. When the force pushes the string through an infinitesimal displacement $\xi(x) \in V_0$, the total *work* performed by $F(x)$ is then the "sum" of the forces acting on the tiny pieces of the string, in other words, the work is the "inner product" of F and ξ,

$$\langle F(x), \xi(x) \rangle = \int_0^L F(x)\xi(x)dx.$$

(Note that the inner product we use here differs from the one used in Section 3.3 by a constant factor.)

On the other hand this work is the amount of potential energy lost when the string undergoes the displacement:

$$\begin{aligned}\langle F(x), \xi(x) \rangle &= \int_0^L \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L \frac{T}{2}\left(\frac{\partial (u+\xi)}{\partial x}\right)^2 dx \\ &= -T\int_0^L \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x}dx + \int_0^L \frac{T}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 dx.\end{aligned}$$

We are imagining that the displacement ξ is infinitesimally small, so terms containing the square of ξ or the square of a derivative of ξ can be ignored, and hence

$$\langle F(x), \xi(x) \rangle = -T\int_0^L \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x}dx.$$

Integration by parts yields

$$\langle F(x), \xi(x) \rangle = T\int_0^L \frac{\partial^2 u}{\partial x^2}\xi(x)dx - T\left(\frac{\partial u}{\partial x}\xi\right)(L) - T\left(\frac{\partial u}{\partial x}\xi\right)(0).$$

Since $\xi(0) = \xi(L) = 0$, we conclude that

$$\int_0^L F(x)\xi(x)dx = \langle F(x), \xi(x) \rangle = T\int_0^L \frac{\partial^2 u}{\partial x^2}\xi(x)dx.$$

Since this formula holds for *all* infinitesimal displacements $\xi(x)$, we must have

$$F(x) = T\frac{\partial^2 u}{\partial x^2},$$

for the force density per unit length.

Now we apply Newton's second law, force = mass $\times$ acceleration, to the function $u(x,t)$. The force acting on a tiny piece of the string of length dx is $F(x)dx$, while the mass of this piece of string is just ρdx, where ρ is the density of the string. Thus Newton's law becomes

$$T\frac{\partial^2 u}{\partial x^2}dx = \rho dx\frac{\partial^2 u}{\partial t^2}.$$

If we divide by ρdx, we obtain the *wave equation*,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}, \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

where $c^2 = T/\rho$.

Just as in the preceding section, we could approximate this partial differential equation by a system of ordinary differential equations. Assume that the string has length $L = 1$ and set $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \text{the displacement of the string at } x_i \text{ at time } t.$$

Then the function $u(x, t)$ can be approximated by the vector-valued function

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix}$$

of one variable, just as before. The wave equation is then approximated by the system of ordinary differential equations

$$\frac{d^2\mathbf{u}}{dt^2} = c^2 n^2 P \mathbf{u},$$

where P is the $(n-1) \times (n-1)$ matrix described in the preceding section. Thus the differential operator

$$\mathbf{L} = \frac{d^2}{dx^2} \quad \text{is approximated by the symmetric matrix} \quad n^2 P,$$

and we expect solutions to the wave equation to behave like solutions to a mechanical system of weights and springs with a large number of degrees of freedom.

Exercises:

4.4.1.a. Show that if $f : \mathsf{R} \to \mathsf{R}$ is any well-behaved function of one variable,

$$u(x, t) = f(x + ct)$$

is a solution to the partial differential equation

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0.$$

(Hint: Use the "chain rule.")

b. Show that if $g : \mathsf{R} \to \mathsf{R}$ is any well-behaved function of one variable,

$$u(x, t) = g(x - ct)$$

is a solution to the partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

c. Show that for any choice of well-behaved functions f and g, the function

$$u(x,t) = f(x+ct) + g(x-ct)$$

is a solution to the differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right]\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0.$$

Remark: This gives a very explicit general solution to the equation for the vibrations of an infinitely long string.

d. Show that

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

is a solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

4.4.2. Show that if the tension and density of a string are given by variable functions $T(x)$ and $\rho(x)$ respectively, then the motion of the string is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho(x)}\frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right).$$

4.5 The initial value problem for the vibrating string

The Fourier sine series can also be used to find the solution to an *initial value problem* for the vibrating string with fixed endpoints at $x = 0$ and $x = L$. We formulate this problem as follows: we seek a function $u(x,t)$, defined for $0 \leq x \leq L$ and $t \geq 0$ such that

1. $u(x,t)$ satisfies the wave equation

   $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.18}$$

   where c is a constant.

2. $u(x,t)$ satisfies the boundary condition $u(0,t) = u(L,t) = 0$, i.e. the displacement of the string is zero at the endpoints.

3. $u(x,t)$ satisfies the initial conditions

   $$u(x,0) = h_1(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h_2(x),$$

   where $h_1(x)$ and $h_2(x)$ are given functions, the initial position and velocity of the string.

Note that the wave equation itself and the boundary condition are *homogeneous* and *linear*, and therefore satisfy the principal of superposition.

Once again, we find the solution to our problem in two steps:

Step I. We find all of the solutions to the homogeneous linear conditions of the special form

$$u(x,t) = f(x)g(t).$$

Step II. We find the superposition of these solution which satisfies the nonhomogeneous initial conditions by means of Fourier analysis.

To carry out Step I, we substitute $u(x,t) = f(x)g(t)$ into the wave equation (4.18) and obtain

$$f(x)g''(t) = c^2 f''(x)g(t).$$

We separate variables, putting all the functions involving t on the left, all the functions involving x on the right:

$$\frac{g''(t)}{g(t)} = c^2 \frac{f''(x)}{f(x)}.$$

Once again, the left-hand side of this equation does not depend on x, while the right-hand side does not depend on t, so neither side can depend upon either x or t. Therefore the two sides must equal a constant λ, and our equation becomes

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda,$$

which separates into two ordinary differential equations,

$$\frac{g''(t)}{c^2 g(t)} = \lambda, \quad \text{or} \quad g''(t) = \lambda c^2 g(t), \tag{4.19}$$

and

$$\frac{f''(x)}{f(x)} = \lambda, \quad \text{or} \quad f''(x) = \lambda f(x). \tag{4.20}$$

Just as in the case of the heat equation, the homogeneous boundary condition $u(0,t) = u(L,t) = 0$ becomes

$$f(0)g(t) = f(L)g(t) = 0,$$

and assuming that $g(t)$ is not identically zero, we obtain

$$f(0) = f(L) = 0.$$

Thus once again we need to find the nontrivial solutions to the boundary value problem,

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), f(0) = 0 = f(L),$$

and just as before, we find that the the only nontrivial solutions are constant multiples of

$$f(x) = \sin(n\pi x/L), \quad \text{with} \quad \lambda = -(n\pi/L)^2, \quad n = 1, 2, 3, \ldots .$$

For each of these solutions, we need to find a corresponding $g(t)$ solving equation (4.19),

$$g''(t) = -(n\pi/L)^2 c^2 g(t), \quad \text{or} \quad g''(t) + (n\pi/L)^2 c^2 g(t) = 0.$$

This is just the equation of simple harmonic motion, and has the general solution

$$g(t) = a\cos(nc\pi t/L) + b\sin(nc\pi t/L),$$

where a and b are constants of integration. Thus we find that the nontrivial product solutions to the wave equation together with the homogeneous boundary condition $u(0,t) = 0 = u(L,t)$ are constant multiples of

$$u_n(x,t) = [a_n\cos(nc\pi t/L) + b_n\sin(nc\pi t/L)]\sin(n\pi x/L).$$

The general solution to the wave equation together with this boundary condition is an arbitrary superposition of these product solutions:

$$u(x,t) = [a_1\cos(c\pi t/L) + b_1\sin(c\pi t/L)]\sin(\pi x/L) \tag{4.21}$$

$$+[a_2\cos(2c\pi t/L) + b_2\sin(2c\pi t/L)]\sin(2\pi x/L) + \ldots . \tag{4.22}$$

The vibration of the string is a superposition of a *fundamental mode* which has frequency

$$\frac{c\pi}{L}\frac{1}{2\pi} = \frac{c}{2L} = \frac{\sqrt{T/\rho}}{2L},$$

and higher modes which have frequencies which are exact integer multiples of this frequency.

Step II consists of determining the constants a_n and b_n in (4.22) so that the initial conditions

$$u(x,0) = h_1(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h_2(x)$$

are satisfied. Setting $t = 0$ in (4.22) yields

$$h_1(x) = u(x,0) = a_1\sin(\pi x/L) + a_2\sin(2\pi x/L) + \ldots ,$$

so we see that the a_n's are the coefficients in the Fourier sine series of h_1.

If we differentiate equation(4.22) with respect to t, we find that

$$\frac{\partial u}{\partial t}(x,t) = [\frac{-c\pi}{L}a_1\sin(c\pi t/L) + \frac{c\pi}{L}b_1\cos(c\pi t/L)]\sin(\pi x/L)$$

$$+\frac{-2c\pi}{L}[a_2\sin(2c\pi t/L)+\frac{c\pi}{L}b_2\sin(2c\pi t/L)]\cos(2\pi x/L)+\dots,$$

and setting $t=0$ yields

$$h_2(x)=\frac{\partial u}{\partial t}(x,0)=\frac{c\pi}{L}b_1\sin(\pi x/L)+\frac{2c\pi}{L}b_2\sin(2\pi x/L)+\dots.$$

We conclude that

$$\frac{nc\pi}{L}b_n=\text{the } n\text{-th coefficient in the Fourier sine series of } h_2(x).$$

Example. Suppose that we want to find the function $u(x,t)$, defined for $0\leq x\leq\pi$ and $t\geq 0$, which satisfies the initial-value problem:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\quad u(0,t)=u(\pi,t)=0,$$

$$u(x,0)=5\sin x+12\sin 2x+6\sin 3x,\quad \frac{\partial u}{\partial t}(x,0)=0.$$

In this case, the first three coefficients for the Fourier sine series of h are

$$a_1=5,\quad a_2=12,\quad a_3=6,$$

and all the others are zero, so the solution must be

$$u(x,t)=5\sin x\cos t+12\sin 2x\cos 2t+6\sin 3x\cos 3t.$$

Exercises:

4.5.1 What happens to the frequency of the fundamental mode of oscillation of a vibrating string when the length of the string is doubled? When the tension on the string is doubled? When the density of the string is doubled?

4.5.2. Find the function $u(x,t)$, defined for $0\leq x\leq\pi$ and $t\geq 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\quad u(0,t)=u(\pi,t)=0,$$

$$u(x,0)=\sin 2x,\quad \frac{\partial u}{\partial t}(x,0)=0.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f''(x)=\lambda f(x),\quad f(0)=0=f(\pi)$$

are

$$\lambda=-n^2,\qquad f(x)=b_n\sin nx,\qquad \text{for } n=1,2,3,\dots,$$

where the b_n's are constants.

4.5.3. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = \sin x + 3\sin 2x - 5\sin 3x, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

4.5.4. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = x(\pi - x), \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

4.5.5. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \sin x + \sin 2x.$$

4.5.6. (For students with access to Mathematica) a. Find the first ten coefficients of the Fourier sine series for

$$h(x) = x - x^4$$

by running the following Mathematica program

```
f[n_] := 2 NIntegrate[(x - x^4) Sin[n Pi x], {x,0,1}];
b = Table[f[n], {n,1,10}]
```

b. Find the first ten terms of the solution to the initial value problem for a vibrating string,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = x - x^4, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

c. Construct a sequence of sketches of the positions of the vibrating string at the times $t_i = ih$, where $h = .1$ by running the Mathematica program:

```
vibstring = Table[
```

```
    Plot[
        Sum[ b[n] Sin[n Pi x] Cos[n Pi t], {n,1,10}],
        {x,0,1}, PlotRange -> {-1,1}
    ], {t,0,1.,.1}
]
```

d. Select the cell containing `vibstring` and animate the sequence of graphics by running "Animate selected graphics," from the Cell menu.

4.6 Heat flow in a circular wire

The theory of Fourier series can also be used to solve the *initial value problem* for the heat equation in a circular wire of radius 1 which is insulated along the sides. In this case, we seek a function $u(\theta, t)$, defined for $\theta \in \mathsf{R}$ and $t \geq 0$ such that

1. $u(\theta, t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}, \tag{4.23}$$

   where c is a constant.

2. $u(\theta, t)$ is periodic of period 2π in the variable θ; in other words,

$$u(\theta + 2\pi, t) = u(\theta, t), \qquad \text{for all } \theta \text{ and } t.$$

3. $u(\theta, t)$ satisfies the initial condition $u(\theta, 0) = h(\theta)$, where $h(\theta)$ is a given function, periodic of period 2π, the initial temperature of the wire.

Once again the heat equation itself and the periodicity condition are homogeneous and linear, so they must be dealt with first. Once we have found the general solution to the homogeneous conditions

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t),$$

we will be able to find the particular solution which satisfies the initial condition

$$u(\theta, 0) = h(\theta)$$

by the theory of Fourier series.

Thus we substitute $u(\theta, t) = f(\theta)g(t)$ into the heat equation (4.23) to obtain

$$f(\theta)g'(t) = c^2 f''(\theta)g(t)$$

and separate variables:

$$\frac{g'(t)}{g(t)} = c^2 \frac{f''(\theta)}{f(\theta)}.$$

The left-hand side of this equation does not depend on θ, while the right-hand side does not depend on t, so neither side can depend upon either θ or t, and we can write

$$\frac{g'(t)}{c^2 g(t)} = \frac{f''(\theta)}{f(\theta)} = \lambda,$$

where λ is a constant. We thus obtain two ordinary differential equations,

$$\frac{g'(t)}{c^2 g(t)} = \lambda, \quad \text{or} \quad g'(t) = \lambda c^2 g(t), \tag{4.24}$$

and

$$\frac{f''(\theta)}{f(\theta)} = \lambda, \quad \text{or} \quad f''(\theta) = \lambda f(\theta). \tag{4.25}$$

The periodicity condition $u(\theta + \pi, t) = u(\theta, t)$ becomes

$$f(\theta + 2\pi)g(t) = f(\theta)g(t),$$

and if $g(t)$ is not identically zero, we must have

$$f(\theta + 2\pi) = f(\theta).$$

Thus to find the nontrivial solutions to the homogeneous linear part of the problem requires us to find the nontrivial solutions to the problem:

$$f''(\theta) = \frac{d^2}{d\theta^2}(f(\theta)) = \lambda f(\theta), \quad f(\theta + 2\pi) = f(\theta). \tag{4.26}$$

We will call (4.11) the *eigenvalue* problem for the differential operator

$$\mathbf{L} = \frac{d^2}{d\theta^2}$$

acting on the space V of functions which are periodic of period 2π.

As before, we need to consider three cases.

Case 1: $\lambda = 0$. In this case, the eigenvalue problem (4.26) becomes

$$f''(\theta) = 0, \quad f(\theta + 2\pi) = f(\theta).$$

The general solution to the differential equation is $f(\theta) = a + b\theta$, and

$$a + b(\theta + 2\pi) = a + b(\theta) \Rightarrow b = 0.$$

Thus the only solution in this case is that where f is constant, and to be consistent with our Fourier series conventions, we write $f(\theta) = a_0/2$, where a_0 is a constant.

Case 2: $\lambda > 0$. In this case, the differential equation

$$f''(\theta) = \lambda f(\theta), \quad \text{or} \quad f''(\theta) - \lambda f(\theta) = 0$$

has the general solution

$$f(\theta) = ae^{(\sqrt{\lambda}\theta)} + be^{-(\sqrt{\lambda}\theta)}.$$

Note that

$$a \neq 0 \Rightarrow f(\theta) \to \pm\infty \quad \text{as} \quad \theta \to \infty,$$

while

$$b \neq 0 \Rightarrow f(\theta) \to \pm\infty \quad \text{as} \quad \theta \to -\infty.$$

Neither of these is consistent with the periodicity conditions $f(\theta + 2\pi) = f(\theta)$, so we conclude that $a = b = 0$, and we obtain no nontrivial solutions in this case.

Case 3: $\lambda < 0$**.** In this case, we set $\omega = \sqrt{-\lambda}$, and rewrite the eigenvalue problem as

$$f''(\theta) + \omega^2 f(\theta) = 0, \quad f(\theta + 2\pi) = f(\theta).$$

We recognize once again our old friend, the differential equation of simple harmonic motion, which has the general solution

$$f(\theta) = a\cos(\omega\theta) + b\sin(\omega\theta) = A\sin(\omega(\theta - \theta_0)).$$

The periodicity condition $f(\theta + 2\pi) = f(\theta)$ implies that $\omega = n$, where n is an integer, which we can assume is positive, and

$$f(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta).$$

Thus we see that the general solution to the eigenvalue problem (4.26) is

$$\lambda = 0 \quad \text{and} \quad f(\theta) = \frac{a_0}{2},$$

or

$$\lambda = -n^2 \quad \text{where } n \text{ is a positive integer and} \quad f(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta).$$

Now we need to find the corresponding solutions to (4.24)

$$g'(t) = \lambda c^2 g(t),$$

for $\lambda = 0, -1, -4, -9, \ldots, -n^2, \ldots$. As before, we find that the solution is

$$g(t) = (\text{constant})e^{-n^2c^2t},$$

where c is a constant. Thus the product solutions to the homogeneous part of the problem are

$$u_0(\theta, t) = \frac{a_0}{2}, \quad u_n(\theta, t) = [a_n\cos(n\theta) + b_n\sin(n\theta)]e^{-n^2c^2t},$$

where $n = 1, 2, 3, \ldots$.

Now we apply the superposition principle—an arbitrary superposition of these product solutions must again be a solution. Thus

$$u(\theta,t)=\frac{a_0}{2}+\Sigma_{n=1}^{\infty}[a_n\cos(n\theta)+b_n\sin(n\theta)]e^{-n^2c^2t} \tag{4.27}$$

is a periodic solution of period 2π to the heat equation (4.23).

To finish the solution to our problem, we must impose the initial condition

$$u(\theta,0)=h(\theta).$$

But setting $t=0$ in (4.27) yields

$$\frac{a_0}{2}+\Sigma_{n=1}^{\infty}[a_n\cos(n\theta)+b_n\sin(n\theta)]=h(\theta),$$

so the constants $a_0, a_1, \dots, b_1, \dots$ are just the Fourier coefficients of $h(\theta)$. Thus the solution to our initial value problem is just (4.27) in which the constants a_k and b_k can be determined via the familiar formulae

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}h(\theta)\cos k\theta d\theta, \qquad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}h(\theta)\sin k\theta d\theta.$$

Note that as $t\to\infty$ the temperature in the circular wire approaches the constant value $a_0/2$.

Exercises:

4.6.1. Find the function $u(\theta,t)$, defined for $0\le\theta\le 2\pi$ and $t\ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial\theta^2}, \quad u(\theta+2\pi,t)=u(\theta,t), \quad u(\theta,0)=2+\sin\theta-\cos 3\theta.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f''(\theta)=\lambda f(\theta), \quad f(\theta+2\pi)=f(\theta)$$

are

$$\lambda=0 \quad \text{and} \quad f(\theta)=\frac{a_0}{2},$$

and

$$\lambda=-n^2 \quad \text{and} \quad f(\theta)=a_n\cos n\theta+b_n\sin n\theta, \qquad \text{for } n=1,2,3,\dots,$$

where the a_n's and b_n's are constants.

4.6.2. Find the function $u(\theta,t)$, defined for $0\le\theta\le 2\pi$ and $t\ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial\theta^2}, \quad u(\theta+2\pi,t)=u(\theta,t),$$

$$u(\theta, 0) = |\theta|, \qquad \text{for } \theta \in [-\pi, \pi].$$

4.6.3. Find the function $u(\theta, t)$, defined for $0 \leq \theta \leq 2\pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t),$$

$$u(\theta, 0) = 2 + \sin\theta - \cos 3\theta, \qquad \frac{\partial u}{\partial t}(\theta, 0) = 0.$$

4.7 Sturm-Liouville Theory*

We would like to be able to analyze heat flow in a bar even if the specific heat $\sigma(x)$, the density $\rho(x)$ and the thermal conductivity $\kappa(x)$ vary from point to point. As we saw in Section 4.1, this leads to consideration of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{\rho(x)\sigma(x)} \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right), \tag{4.28}$$

where we make the standing assumption that $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are positive.

We imagine that the bar is situated along the x-axis with its endpoints situated at $x = a$ and $x = b$. As in the constant coefficient case, we expect that there should exist a unique function $u(x, t)$, defined for $a \leq x \leq b$ and $t \geq 0$ such that

1. $u(x, t)$ satisfies the heat equation (4.28).
2. $u(x, t)$ satisfies the boundary condition $u(a, t) = u(b, t) = 0$.
3. $u(x, t)$ satisfies the initial condition $u(x, 0) = h(x)$, where $h(x)$ is a given function, defined for $x \in [a, b]$, the initial temperature of the bar.

Just as before, we substitute $u(x, t) = f(x)g(t)$ into (4.28) and obtain

$$f(x)g'(t) = \frac{1}{\rho(x)\sigma(x)} \frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right) g(t).$$

Once again, we separate variables, putting all the functions involving t on the left, all the functions involving x on the right:

$$\frac{g'(t)}{g(t)} = \frac{1}{\rho(x)\sigma(x)} \frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right) \frac{1}{f(x)}.$$

As usual, the two sides must equal a constant, which we denote by λ, and our equation separates into two ordinary differential equations,

$$g'(t) = \lambda g(t), \tag{4.29}$$

and

$$\frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right) = \lambda f(x). \tag{4.30}$$

Under the assumption that u is not identically zero, the boundary condition $u(a,t) = u(b,t) = 0$ yields

$$f(a) = f(b) = 0.$$

Thus to find the nontrivial solutions to the homogeneous linear part of the problem, we need to find the nontrivial solutions to the boundary value problem:

$$\frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right) = \lambda f(x), \qquad f(a) = 0 = f(b). \tag{4.31}$$

We call this the *eigenvalue problem* or *Sturm-Liouville problem* for the differential operator

$$\mathbf{L} = \frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{d}{dx}\right),$$

which acts on the space V_0 of well-behaved functions $f : [a,b] \to \mathsf{R}$ which vanish at the endpoints a and b. The *eigenvalues* of $\mathbf{L}$ are the constants λ for which (4.31) has nontrivial solutions. Given an eigenvalue λ, the corresponding *eigenspace* is

$$W_\lambda = \{f \in V_0 : \ f \text{ satisfies } (4.31)\}.$$

Nonzero elements of the eigenspaces are called *eigenfunctions*.

If the functions $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are complicated, it may be impossible to solve this eigenvalue problem explicitly, and one may need to employ numerical methods to obtain approximate solutions. Nevertheless, it is reassuring to know that the theory is quite parallel to the constant coefficient case that we treated in previous sections. The following theorem, due to the nineteenth century mathematicians Sturm and Liouville, is proven in more advanced texts:[4]

Theorem. *Suppose that $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are smooth functions which are positive on the interval $[a,b]$. Then all of the eigenvalues of $\mathbf{L}$ are negative real numbers, and each eigenspace is one-dimensional. Moreover, the eigenvalues can be arranged in a sequence*

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots,$$

with $\lambda_n \to -\infty$. Finally, every well-behaved function can be represented on $[a,b]$ as a convergent sum of eigenfunctions.

Suppose that $f_1(x)$, $f_2(x)$, ..., $f_n(x)$, ... are eigenfunctions corresponding to the eigenvalues λ_1, λ_2, ..., λ_n, ... Then the general solution to the heat equation (4.28) together with the boundary conditions $u(a,t) = u(b,t) = 0$ is

$$u(x,t) = \sum_{n=0}^{\infty} c_n f_n(x) e^{-\lambda_n t},$$

[4]For further discussion, see Boyce and DiPrima, *Elementary differential equations and boundary value problems*, Seventh edition, Wiley, New York, 2001.

where the c_n's are arbitrary constants.

To determine the c_n's in terms of the initial temperature $h(x)$, we need a generalization of the theory of Fourier series. The key idea here is that the eigenspaces should be orthogonal with respect to an appropriate inner product. The inner product should be one which makes $\mathbf{L}$ like a symmetric matrix. To arrange this, the inner product that we need to use on V_0 is the one defined by the formula

$$\langle f, g \rangle = \int_a^b \rho(x)\sigma(x)f(x)g(x)dx.$$

Lemma. *With respect to this inner product, eigenfunctions corresponding to distinct eigenvalues are perpendicular.*

The proof hinges on the fact that

$$\langle \mathbf{L}(f), g \rangle = \langle f, \mathbf{L}(g) \rangle, \qquad \text{for } f, g \in V_0,$$

so that if we thought of L as represented by a matrix, the matrix would be symmetric. This identity can be verified by integration by parts; indeed,

$$\langle \mathbf{L}(f), g \rangle = \int_a^b \rho(x)\sigma(x)\mathbf{L}(f)(x)g(x)dx = \int_a^b \frac{d}{dx}\left(K(x)\frac{df}{dx}(x)\right)g(x)$$

$$= -\int_a^b K(x)\frac{df}{dx}(x)\frac{dg}{dx}(x)dx = \cdots = \langle f, \mathbf{L}(g) \rangle,$$

where the steps represented by dots are just like the first steps, but in reverse order.

It follows that if $f_i(x)$ and $f_j(x)$ are eigenfunctions corresponding to distinct eigenvalues λ_i and λ_j, then

$$\lambda_i \langle f_i, f_j \rangle = \langle \mathbf{L}(f), g \rangle = \langle f, \mathbf{L}(g) \rangle = \lambda_j \langle f_i, f_j \rangle,$$

and hence

$$(\lambda_i - \lambda_j)\langle f_i, f_j \rangle = 0.$$

Since $\lambda_i - \lambda_j \neq 0$, we conclude that f_i and f_j are perpendicular with respect to the inner product $\langle \cdot, \cdot \rangle$, as claimed.

Thus to determine the c_n's, we can use exactly the same orthogonality techniques that we have used before. Namely, if we normalize the eigenfunctions so that they have unit length and are orthogonal to each other with respect to $\langle, \cdot, \cdot \rangle$, then

$$c_n = \langle h, f_n \rangle,$$

or equivalently, c_n is just the projection of h in the f_n direction.

Example. We consider the operator

$$\mathbf{L} = x\frac{d}{dx}\left(x\frac{d}{dx}\right),$$

which acts on the space V_0 of functions $f : [1, e^\pi] \to \mathsf{R}$ which vanish at the endpoints of the interval $[1, e^\pi]$. To solve the eigenvalue problem, we need to find the nontrivial solutions to

$$x\frac{d}{dx}\left(x\frac{df}{dx}(x)\right) = \lambda f(x), \qquad f(1) = 0 = f(e^\pi). \tag{4.32}$$

We could find these nontrivial solutions by using the techniques we have learned for treating Cauchy-Euler equations.

However, there is a simpler approach, based upon the technique of substitution. Namely, we make a change of variables $x = e^z$ and note that since

$$dx = e^z dz, \qquad \frac{d}{dx} = \frac{1}{e^z}\frac{d}{dz} \qquad \text{and hence} \qquad x\frac{d}{dx} = e^z\frac{1}{e^z}\frac{d}{dz} = \frac{d}{dz}.$$

Thus if we set

$$\tilde{f}(z) = f(x) = f(e^z),$$

the eigenvalue problem (4.32) becomes

$$\frac{d^2\tilde{f}}{dz^2}(z) = \lambda\tilde{f}(z), \qquad \tilde{f}(0) = 0 = \tilde{f}(\pi),$$

a problem which we have already solved. The nontrivial solutions are

$$\lambda_n = -n^2, \qquad \tilde{f}_n(z) = \sin nz, \qquad \text{where } n = 1, 2, \ldots.$$

Thus the eigenvalues for our original problem are

$$\lambda_n = -n^2, \qquad \text{for } n = 1, 2, \ldots,$$

and as corresponding eigenfunctions we can take

$$f_n(x) = \sin(n\log x),$$

where log denotes the natural or base e logarithm. The lemma implies that these eigenfunctions will be perpendicular with respect to the inner product $\langle\cdot,\cdot\rangle$, defined by

$$\langle f, g\rangle = \int_1^{e^\pi}\frac{1}{x}f(x)g(x)dx. \tag{4.33}$$

Exercises:

4.7.1. Show by direct integration that if $m \neq n$, the functions

$$f_m(x) = \sin(m\log x) \qquad \text{and} \qquad f_n(x) = \sin(n\log x)$$

are perpendicular with respect to the inner product defined by (4.33).

4.7.2. Find the function $u(x,t)$, defined for $1 \leq x \leq e^\pi$ and $t \geq 0$, which satisfies the following initial-value problem for a heat equation with variable coefficients:

$$\frac{\partial u}{\partial t} = x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), \quad u(1,t) = u(e^\pi, t) = 0,$$

$$u(x,0) = 3\sin(\log x) + 7\sin(2\log x) - 2\sin(3\log x).$$

4.7.3.a. Find the solution to the eigenvalue problem for the operator

$$\mathbf{L} = x\frac{d}{dx}\left(x\frac{d}{dx}\right) - 3,$$

which acts on the space V_0 of functions $f : [1, e^\pi] \to \mathsf{R}$ which vanish at the endpoints of the interval $[1, e^\pi]$.

b. Find the function $u(x,t)$, defined for $1 \leq x \leq e^\pi$ and $t \geq 0$, which satisfies the following initial-value problem for a heat equation with variable coefficients:

$$\frac{\partial u}{\partial t} = x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - 3u, \quad u(1,t) = u(e^\pi, t) = 0,$$

$$u(x,0) = 3\sin(\log x) + 7\sin(2\log x) - 2\sin(3\log x).$$

4.8 Numerical solutions to the eigenvalue problem*

We can also apply Sturm-Liouville theory to study the motion of a string of variable mass density. We can imagine a violin string stretched out along the x-axis with endpoints at $x = 0$ and $x = 1$ covered with a layer of varnish which causes its mass density to vary from point to point. We could let

$$\rho(x) = \text{the mass density of the string at } x \text{ for } 0 \leq x \leq 1.$$

If the string is under constant tension T, its motion might be governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho(x)}\frac{\partial^2 u}{\partial x^2}, \tag{4.34}$$

which would be subject to the Dirichlet boundary conditions

$$u(0,t) = 0 = u(1,t), \qquad \text{for all } t \geq 0. \tag{4.35}$$

It is natural to try to find the general solution to (4.34) and (4.35) by separation of variables, letting $u(x,t) = f(x)g(t)$ as usual. Substituting into (4.34) yields

$$f(x)g''(t) = \frac{T}{\rho(x)}f''(x)g(t), \quad \text{or} \quad \frac{g''(t)}{g(t)} = \frac{T}{\rho(x)}\frac{f''(x)}{f(x)}.$$

The two sides must equal a constant, denoted by λ, and the partial differential equation separates into two ordinary differential equations,

$$\frac{T}{\rho(x)} f''(x) = \lambda f(x), \qquad g''(t) = \lambda g(t).$$

The Dirichlet boundary conditions (4.35) yield $f(0) = 0 = f(1)$. Thus f must satisfy the eigenvalue problem

$$\mathbf{L}(f) = \lambda f, \quad f(0) = 0 = f(1), \qquad \text{where} \qquad \mathbf{L} = \frac{T}{\rho(x)} \frac{d^2}{dx^2}.$$

Although the names of the functions appearing in $\mathbf{L}$ are a little different than those used in the previous section, the same Theorem applies. Thus the eigenvalues of $\mathbf{L}$ are negative real numbers and each eigenspace is one-dimensional. Moreover, the eigenvalues can be arranged in a sequence

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots,$$

with $\lambda_n \to -\infty$. Finally, every well-behaved function can be represented on $[a, b]$ as a convergent sum of eigenfunctions. If $f_1(x)$, $f_2(x)$, ... , $f_n(x)$, ... are eigenfunctions corresponding to the eigenvalues λ_1, λ_2, ... , λ_n, ... Then the general solution to (4.34) and (4.35) is

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \left[a_n \cos(\sqrt{-\lambda_n} t) + b_n \sin(\sqrt{-\lambda_n} t) \right],$$

where the a_n's and b_n's are arbitrary constants. Each term in this sum represents one of the modes of oscillation of the vibrating string.

In constrast to the case of constant density, it is usually not possible to find simple explicit eigenfunctions when the density varies. It is therefore usually necessary to use numerical methods.

The simplest numerical method is the one outlined in §4.3. For $0 \leq i \leq n$, we let $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \text{the displacement at } x_i \text{ at time } t.$$

Since $u_0(t) = 0 = u_n(t)$ by the boundary conditions, the displacement at time t is approximated by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix},$$

a vector-valued function of one variable. The partial derivative

$$\frac{\partial^2 u}{\partial t^2} \qquad \text{is approximated by} \qquad \frac{d^2 \mathbf{u}}{dt^2},$$

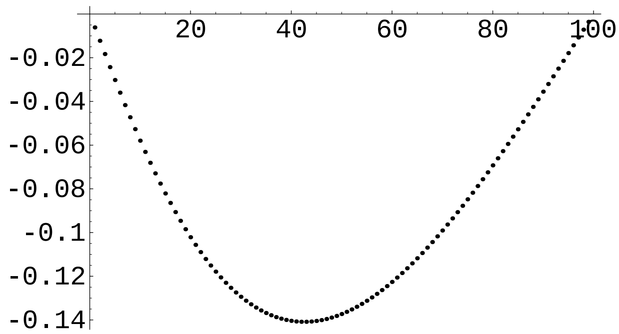

Figure 4.1: Shape of the lowest mode when $\rho = 1/(x + .1)$.

and as we saw in §4.3, the partial derivative

$$\frac{\partial^2 u}{\partial x^2} \qquad \text{is approximated by} \qquad n^2 P\mathbf{u},$$

where

$$P = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -2 \end{pmatrix}.$$

Finally, the coefficient $(T/\rho(x))$ can be represented by the diagonal matrix

$$Q = \begin{pmatrix} T/\rho(x_1) & 0 & 0 & \cdots & 0 \\ 0 & T/\rho(x_2) & 0 & \cdots & 0 \\ 0 & 0 & T/\rho(x_3) & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & T/\rho(x_{n-1}) \end{pmatrix}.$$

Putting all this together, we find that our wave equation with variable mass density is approximated by a second order homogeneous linear system of ordinary differential equations

$$\frac{d^2\mathbf{u}}{dt^2} = A\mathbf{u}, \qquad \text{where} \qquad A = n^2 QP.$$

The eigenvalues of low absolute value are approximated by the eigenvalues of A, while the eigenfunctions representing the lowest frequency modes of oscillation are approximated eigenvectors corresponding to the lowest eigenvalues of A.

For example, we could ask the question: What is the shape of the lowest mode of oscillation in the case where $\rho(x) = 1/(x+.1)$? To answer this question, we could utilize the following Mathematica program (which is quite similar to the one presented in Exercise 2.5.2):

```
n := 100; rho[x_] := 1/(x + .1);
m := Table[Max[2-Abs[i-j],0], { i,n-1 } ,{ j,n-1 } ];
p := m - 4 IdentityMatrix[n-1];
q := DiagonalMatrix[Table[(1/rho[i/n]), { i,1,n-1 } ]];
a := n∧2 q.p; eigenvec = Eigenvectors[N[a]];
ListPlot[eigenvec[[n-1]]]
```

If we run this program we obtain a graph of the shape of the lowest mode, as shown in Figure 4.1. Note that instead of approximating a sine curve, our numerical approximation to the lowest mode tilts somewhat to the left.

Chapter 5

PDE's in Higher Dimensions

5.1 The three most important linear partial differential equations

In higher dimensions, the three most important linear partial differential equations are Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

the heat equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

and the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where c is a nonzero constant. Techniques developed for studying these equations can often be applied to closely related equations.

Each of these three equations is *homogeneous linear*, that is each term contains u or one of its derivatives to the first power. This ensures that the principle of superposition will hold,

$$u_1 \text{ and } u_2 \text{ solutions} \quad \Rightarrow \quad c_1 u_1 + c_2 u_2 \text{ is a solution,}$$

for any choice of constants c_1 and c_2. The principle of superposition is essential if we want to apply separation of variables and Fourier analysis techniques.

In the first few sections of this chapter, we will derive these partial differential equations in several physical contexts. We will begin by using the divergence theorem to derive the heat equation, which in turn reduces in the steady-state case to Laplace's equation. We then present two derivations of the wave equation, one for vibrating membranes and one for sound waves. Exactly the same wave equation also describes electromagnetic waves, gravitational waves, or water waves in a linear approximation. It is remarkable that the principles developed to solve the three basic linear partial differential equations can be applied in so many contexts.

In a few cases, it is possible to find explicit solutions to these partial differential equations under the simplest boundary conditions. For example, the general solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for the vibrations of an infinitely long string, is

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are arbitrary well-behaved functions of a single variable.

Slightly more complicated cases require the technique of "separation of variables" together with Fourier analysis, as we studied before. Separation of variables reduces these partial differential equations to linear ordinary differential equations, often with variable coefficients. For example, to find the explicit solution to the heat equation in a circular room, we will see that it is necessary to solve Bessel's equation.

The most complicated cases cannot be solved by separation of variables, and one must resort to numerical methods, together with sufficient theory to understand the qualitative behaviour of the solutions.

In the rest of this section, we consider our first example, the equation governing heat flow through a region of (x,y,z)-space, under the assumption that no heat is being created or destroyed within the region. Let

$$u(x,y,z,t) = \text{temperature at } (x,y,z) \text{ at time } t.$$

If $\sigma(x,y,z)$ is the specific heat at the point (x,y,z) and $\rho(x,y,z)$ is the density of the medium at (x,y,z), then the heat within a given region D in (x,y,z)-space is given by the formula

$$\text{Heat within } D = \int\int\int_D \rho(x,y,z)\sigma(x,y,z)u(x,y,z,t)dxdydz.$$

If no heat is being created or destroyed within D, then by conservation of energy, the rate at which heat leaves D equals minus the rate of change of heat within D, which is

$$-\frac{d}{dt}\left[\int\int\int_D \rho\sigma u dxdydz\right] = -\int\int\int_D \rho\sigma\frac{\partial u}{\partial t}dxdydz, \tag{5.1}$$

by differentiating under the integral sign.

On the other hand, heat flow can be represented by a vector field $\mathbf{F}(x, y, z, t)$ which points in the direction of greatest decrease of temperature,

$$\mathbf{F}(x, y, z, t) = -\kappa(x, y, z)(\nabla u)(x, y, z, t),$$

where $\kappa(x, y, z)$ is the so-called *thermal conductivity* of the medium at (x, y, z). Thus the rate at which heat leaves the region D through a small region in its boundary of area dA is

$$-(\kappa \nabla u) \cdot \mathbf{N} dA,$$

where $\mathbf{N}$ is the unit normal which points out of D. The total rate at which heat leaves D is given by the flux integral

$$-\int\int_{\partial \mathbf{D}} (\kappa \nabla u) \cdot \mathbf{N} dA,$$

where ∂D is the surface bounding D. It follows from the divergence theorem that

$$\text{Rate at which heat leaves } D = -\int\int\int_D \nabla \cdot (\kappa \nabla u) dxdydz. \tag{5.2}$$

From formulae (5.1) and (5.2), we conclude that

$$\int\int\int_D \rho\sigma \frac{\partial u}{\partial t} dxdydz = \int\int\int_D \nabla \cdot (\kappa \nabla u) dxdydz.$$

This equation is true for *all* choices of the region D, so the integrands on the two sides must be equal:

$$\rho(x, y, z)\sigma(x, y, z) \frac{\partial u}{\partial t}(x, y, z, t) = \nabla \cdot (\kappa \nabla u)(x, y, z, t).$$

Thus we finally obtain the *heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{\rho(x, y, z)\sigma(x, y, z)} \nabla \cdot (\kappa(x, y, z)(\nabla u)).$$

In the special case where the region D is *homogeneous*, i.e. its properties are the same at every point, $\rho(x, y, z)$, $\sigma(x, y, z)$ and $\kappa(x, y, z)$ are constants, and the heat equation becomes

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho\sigma} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right].$$

If we wait long enough, so that the temperature is no longer changing, the "steady-state" temperature $u(x, y, z)$ must satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

If the temperature is independent of z, the function $u(x, y) = u(x, y, z)$ must satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Exercises:

5.1.1. For which of the following differential equations is it true that the superposition principle holds?

$$\text{a.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

$$\text{b.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + u = 0.$$

$$\text{c.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + u^2 = 0.$$

$$\text{d.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = e^x.$$

$$\text{e.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = e^x u.$$

Explain your answers.

5.1.2. Suppose that a chemical reaction creates heat at the rate

$$\lambda(x, y, z)u(x, y, z, t) + \nu(x, y, z),$$

per unit volume. Show that in this case the equation governing heat flow is

$$\rho(x, y, z)\sigma(x, y, z)\frac{\partial u}{\partial t} = \lambda(x, y, z)u(x, y, z, t) + \nu(x, y, z) + \nabla \cdot (\kappa(x, y, z)(\nabla u)).$$

5.2 The Dirichlet problem

The reader will recall that the space of solutions to a homogeneous linear second order *ordinary* differential equation, such as

$$\frac{d^2 u}{dt^2} + p\frac{du}{dt} + qu = 0$$

is two-dimensional, a particular solution being determined by two constants. By contrast, the space of solutions to Laplace's partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5.3}$$

is *infinite*-dimensional. For example, the function $u(x, y) = x^3 - 3xy^2$ is a solution to Laplace's equation, because

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x,$$

$$\frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x,$$

and hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

Similarly,

$$u(x, y) = 7, \quad u(x, y) = x^4 - 6x^2y^2 + y^4, \quad \text{and} \quad u(x, y) = e^x \sin y$$

are solutions to Laplace's equation. A solution to Laplace's equation is called a *harmonic* function. It is not difficult to construct infinitely many linearly independent harmonic functions of two variables.

Thus to pick out a particular solution to Laplace's equation, we need boundary conditions which will impose infinitely many constraints. To see what boundary conditions are natural to impose, we need to think of a physical problem which leads to Laplace's equation. Suppose that $u(x, y)$ represents the steady-state distribution of temperature throughout a uniform slab in the shape of a region D in the (x, y)-plane. If we specify the temperature on the boundary of the region, say by setting up heaters and refrigerators controlled by thermostats along the boundary, we might expect that the temperature inside the room would be uniquely determined. We need infinitely many heaters and refrigerators because there are infinitely many points on the boundary. Specifying the temperature at each point of the boundary imposes infinitely many constraints on the harmonic function which realizes the steady-state temperature within D.

The Dirichlet Problem for Laplace's Equation. *Let D be a bounded region in the (x, y)-plane which is bounded by a curve ∂D, and let $\phi : \partial D \to \mathsf{R}$ be a continuous function. Find a harmonic function $u : D \to \mathsf{R}$ such that*

$$u(x, y) = \phi(x, y), \qquad \text{for } (x, y) \in \partial D.$$

Our physical intuition suggests that the Dirichlet problem will always have a unique solution. This is proven mathematically for many choices of boundary in more advanced texts on complex variables and partial differential equations. The mathematical proof that a unique solution exists provides evidence that the mathematical model we have constructed for heat flow may in fact be valid.

Our goal here is to find the explicit solutions in the case where the region D is sufficiently simple. Suppose, for example, that

$$D = \{(x, y) \in \mathsf{R}^2 : 0 \le x \le a, 0 \le y \le b\}.$$

Suppose, moreover that the function $\phi : \partial D \to \mathsf{R}$ vanishes on three sides of ∂D, so that

$$\phi(0, y) = \phi(a, y) = \phi(x, 0) = 0,$$

while

$$\phi(x, b) = f(x),$$

where $f(x)$ is a given continuous function which vanishes when $x = 0$ and $x = a$.

In this case, we seek a function $u(x, y)$, defined for $0 \le x \le a$ and $0 \le y \le b$, such that

1. $u(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{5.4}$$

2. $u(x, y)$ satisfies the homogeneous boundary conditions $u(0, y) = u(a, y) = u(x, 0) = 0$.

3. $u(x, y)$ satisfies the nonhomogeneous boundary condition $u(x, b) = f(x)$, where $f(x)$ is a given function.

Note that the Laplace equation itself and the homogeneous boundary conditions satisfy the superposition principle—this means that if u_1 and u_2 satisfy these conditions, so does $c_1 u_1 + c_2 u_2$, for any choice of constants c_1 and c_2.

Our method for solving the Dirichlet problem consists of two steps:

Step I. We find all of the solutions to Laplace's equation together with the homogeneous boundary conditions which are of the special form

$$u(x, y) = X(x)Y(y).$$

By the superposition principle, an arbitrary linear superposition of these solutions will still be a solution.

Step II. We find the particular solution which satisfies the nonhomogeneous boundary condition by Fourier analysis.

To carry out Step I, we substitute $u(x, y) = X(x)Y(y)$ into Laplace's equation (5.3) and obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Next we separate variables, putting all the functions involving x on the left, all the functions involving y on the right:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left-hand side of this equation does not depend on y, while the right-hand side does not depend on x. Hence neither side can depend upon either x or y.

In other words, the two sides must equal a constant, which we denote by λ, and call the separating constant, as before. Our equation now becomes

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

which separates into two ordinary differential equations,

$$X''(x) = \lambda X(x), \tag{5.5}$$

and

$$Y''(y) = -\lambda Y(y). \tag{5.6}$$

The homogeneous boundary condition $u(0, y) = u(a, y) = 0$ imply that

$$X(0)Y(y) = X(a)Y(y) = 0.$$

If $Y(y)$ is not identically zero,

$$X(0) = X(a) = 0.$$

Thus we need to find the nontrivial solutions to a boundary value problem for an ordinary differential equation:

$$X''(x) = \frac{d^2}{dx^2}(X(x)) = \lambda X(x), \quad X(0) = 0 = X(a),$$

which we recognize as the eigenvalue problem for the differential operator

$$L = \frac{d^2}{dx^2}$$

acting on the space V_0 of functions which vanish at 0 and a. We have seen before that the only nontrivial solutions to equation (5.5) are constant multiples of

$$X(x) = \sin(n\pi x/a), \quad \text{with} \quad \lambda = -(n\pi/a)^2, \quad n = 1, 2, 3, \ldots .$$

For each of these solutions, we need to find a corresponding $Y(y)$ solving equation (5.6),

$$Y''(y) = -\lambda Y(y),$$

where $\lambda = -(n\pi/a)^2$, together with the boundary condition $Y(0) = 0$. The differential equation has the general solution

$$Y(y) = A\cosh(n\pi y/a) + B\sinh(n\pi y/a),$$

where A and B are constants of integration, and the boundary condition $Y(0) = 0$ implies that $A = 0$. Thus we find that the nontrivial product solutions to Laplace's equation together with the homogeneous boundary conditions are constant multiples of

$$u_n(x, y) = \sin(n\pi x/a)\sinh(n\pi y/a).$$

The *general solution* to Laplace's equation with these boundary conditions is a general superposition of these product solutions:

$$u(x,y) = B_1 \sin(\pi x/a)\sinh(\pi y/a) + B_2 \sin(2\pi x/a)\sinh(2\pi y/a) + \dots . \tag{5.7}$$

To carry out Step II, we need to determine the constants $B_1, B_2, \dots$ which occur in (5.7) so that

$$u(x,b) = f(x).$$

Substitution of $y = b$ into (5.7) yields

$$f(x) = B_1 \sin(\pi x/a)\sinh(\pi b/a) + B_2 \sin(2\pi x/a)\sinh(2\pi b/a) +$$

$$\dots + B_k \sin(2\pi k/a)\sinh(k\pi b/a) + \dots .$$

We see that $B_k \sinh(k\pi b/a)$ is the k-th coefficient in the Fourier sine series for $f(x)$.

For example, if $a = b = \pi$, and $f(x) = 3\sin x + 7\sin 2x$, then we must have

$$f(x) = B_1 \sin(x)\sinh(\pi) + B_2 \sin(2x)\sinh(2\pi),$$

and hence

$$B_1 = \frac{3}{\sinh(\pi)}, \quad B_2 = \frac{7}{\sinh(2\pi)}, \quad B_k = 0 \quad \text{for} \quad k = 3, \dots .$$

Thus the solution to Dirichlet's problem in this case is

$$u(x,y) = \frac{3}{\sinh(\pi)} \sin x \sinh y + \frac{7}{\sinh(2\pi)} \sin 2x \sinh 2y.$$

Exercises:

5.2.1. Which of the following functions are harmonic?

a. $f(x,y) = x^2 + y^2$.

b. $f(x,y) = x^2 - y^2$.

c. $f(x,y) = e^x \cos y$.

d. $f(x,y) = x^3 - 3xy^2$.

5.2.2. a. Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x,y), 0 \le x \le \pi, 0 \le y \le \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = u(\pi,y) = 0,$$

$$u(x,0) = 0, \qquad u(x,\pi) = \sin x - 2\sin 2x + 3\sin 3x.$$

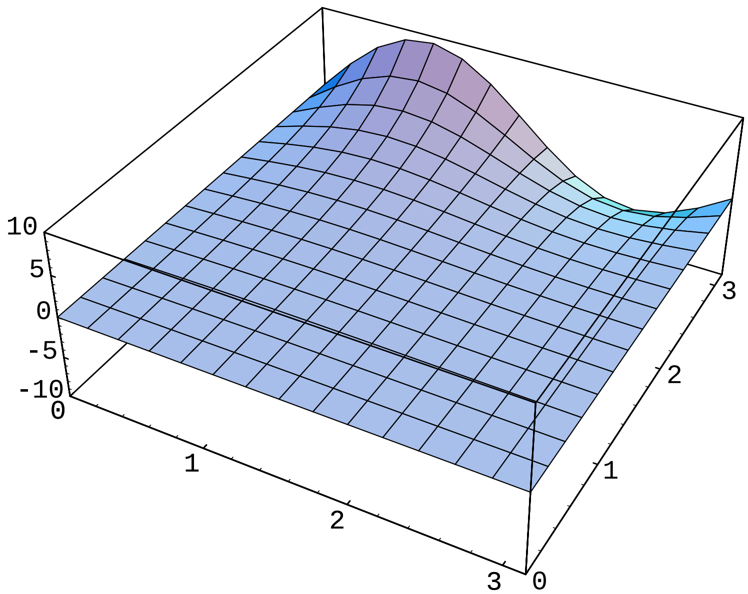

Figure 5.1: Graph of $u(x,y) = \frac{3}{\sinh(\pi)} \sin x \sinh y + \frac{7}{\sinh(2\pi)} \sin 2x \sinh 2y$.

b. Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x,y), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = 0,$$

$$u(\pi,y) = \sin 2y + 3\sin 4y, \qquad u(x,0) = 0 = u(x,\pi).$$

c. By adding the solutions to parts a and c together, find the solution to the Dirichlet problem: Find $u(x,y), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = 0,$$

$$u(\pi,y) = \sin 2y + 3\sin 4y, \qquad u(x,0) = 0, \qquad u(x,\pi) = \sin x - 2\sin 2x + 3\sin 3x.$$

5.3 Initial value problems for heat equations

The physical interpretation behind the heat equation suggests that the following initial value problem should have a unique solution:

Let D be a bounded region in the (x,y)-plane which is bounded by a piecewise smooth curve ∂D, and let $h : D \to \mathsf{R}$ be a continuous function which vanishes on ∂D. Find a function $u(x,y,t)$ such that

1. u satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{5.8}$$

2. u satisfies the "Dirichlet boundary condition" $u(x, y, t) = 0$, for $(x, y) \in \partial D$.

3. u satisfies the initial condition $u(x, y, 0) = h(x, y)$.

The first two of these conditions are homogeneous and linear, so our strategy is to treat them first by separation of variables, and then use Fourier analysis to satisfy the last condition.

In this section, we consider the special case where

$$D = \{(x, y) \in \mathsf{R}^2 : 0 \le x \le a, 0 \le y \le b\},$$

so that the Dirichlet boundary condition becomes

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0.$$

In this case, separation of variables is done in two stages. First, we write

$$u(x, y, t) = f(x, y)g(t),$$

and substitute into (5.8) to obtain

$$f(x, y)g'(t) = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) g(t).$$

Then we divide by $c^2 f(x, y)g(t)$,

$$\frac{1}{c^2 g(t)} g'(t) = \frac{1}{f(x, y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

The left-hand side of this equation does not depend on x or y while the right hand side does not depend on t. Hence neither side can depend on x, y, or t, so both sides must be constant. If we let λ denote the constant, we obtain

$$\frac{1}{c^2 g(t)} g'(t) = \lambda = \frac{1}{f(x, y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

This separates into an ordinary differential equation

$$g'(t) = c^2 \lambda g(t), \tag{5.9}$$

and a partial differential equation

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = \lambda f, \tag{5.10}$$

called the *Helmholtz equation*. The Dirichlet boundary condition yields

$$f(0,y) = f(a,y) = f(x,0) = f(x,b) = 0.$$

In the second stage of separation of variables, we set $f(x,y) = X(x)Y(y)$ and substitute into (5.10) to obtain

$$X''(x)Y(y) + X(x)Y''(y) = \lambda X(x)Y(y),$$

which yields

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \lambda,$$

or

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}.$$

The left-hand side depends only on x, while the right-hand side depends only on y, so both sides must be constant,

$$\frac{X''(x)}{X(x)} = \mu = \lambda - \frac{Y''(y)}{Y(y)}.$$

Hence the Helmholtz equation divides into two ordinary differential equations

$$X''(x) = \mu X(x), \quad Y''(y) = \nu Y(y), \quad \text{where} \quad \mu + \nu = \lambda.$$

The "Dirichlet boundary conditions" now become conditions on $X(x)$ and $Y(y)$:

$$X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0.$$

The only nontrivial solutions are

$$X(x) = \sin(m\pi x/a), \quad \text{with} \quad \mu = -(m\pi/a)^2, \quad m = 1, 2, 3, \ldots,$$

and

$$Y(y) = \sin(n\pi y/b), \quad \text{with} \quad \nu = -(n\pi/b)^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding solutions of the Helmholtz equation are

$$f_{mn}(x,y) = \sin(m\pi x/a)\sin(n\pi y/b), \quad \text{with} \quad \lambda_{mn} = -(m\pi/a)^2 - (n\pi/b)^2.$$

For any given choice of m and n, the corresponding solution to (5.9) is

$$g(t) = e^{-c^2\lambda_{mn}t} = e^{-c^2((m\pi/a)^2+(n\pi/b)^2)t}.$$

Hence for each choice of m and n, we obtain a product solution to the heat equation with Dirichlet boundary condition:

$$u_{m,n}(x,y,t) = \sin(m\pi x/a)\sin(n\pi y/b)e^{-c^2((m\pi/a)^2+(n\pi/b)^2)t}.$$

The general solution to the heat equation with Dirichlet boundary conditions is an arbitrary superposition of these product solutions,

$$u(x,y,t) = \sum_{m,n=1}^{\infty} b_{mn} \sin(m\pi x/a)\sin(n\pi y/b)e^{-c^2((m\pi/a)^2+(n\pi/b)^2)t}. \qquad (5.11)$$

To find the constants b_{mn} appearing in (5.11), we need to apply the initial condition $u(x,y,0) = h(x,y)$. The result is

$$h(x,y) = \sum_{m,n=1}^{\infty} b_{mn} \sin(m\pi x/a)\sin(n\pi y/b), \qquad (5.12)$$

expressing the fact that the b_{mn}'s are the coefficients of what is called the *double Fourier sine series* of h. As in the case of ordinary Fourier sine series, the b_{mn}'s can be determined by an explicit formula. To determine it, we multiply both sides of (5.12) by $(2/a)\sin(p\pi x/a)$, where p is a positive integer, and integrate with respect to x to obtain

$$\frac{2}{a}\int_0^a h(x,y)\sin(p\pi x/a)dx$$

$$= \sum_{m,n=1}^{\infty} b_{mn}\left[\frac{2}{a}\int_0^a \sin(p\pi x/a)\sin(m\pi x/a)dx\right]\sin(n\pi y/b).$$

The expression within brackets is one if $p = m$ and otherwise zero, so

$$\frac{2}{a}\int_0^a h(x,y)\sin(p\pi x/a)dx = \sum_{n=1}^{\infty} b_{pn}\sin(n\pi y/b).$$

Next we multiply by $(2/b)\sin(q\pi y/b)$, where q is a positive integer, and integrate with respect to y to obtain

$$\frac{2}{a}\frac{2}{b}\int_0^a\int_0^b h(x,y)\sin(p\pi x/a)\sin(q\pi y/b)dxdy$$

$$= \sum_{n=1}^{\infty} b_{pn}\left[\frac{2}{b}\int_0^b \sin(n\pi y/b)\sin(q\pi y/b)dy\right].$$

The expression within brackets is one if $q = n$ and otherwise zero, so we finally obtain

$$b_{pq} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b h(x,y)\sin(p\pi x/a)\sin(q\pi y/b)dxdy. \qquad (5.13)$$

Suppose, for example, that $c = 1$, $a = b = \pi$ and

$$h(x,y) = \sin x\sin y + 3\sin 2x\sin y + 7\sin 3x\sin 2y.$$

In this case, we do not need to carry out the integration indicated in (5.13) because comparison with (5.12) shows that

$$b_{11} = 1, \quad b_{21} = 3, \quad b_{32} = 7,$$

and all the other b_{mn}'s must be zero. Thus the solution to the initial value problem in this case is

$$u(x, y, t) = \sin x \sin y e^{-2t} + 3 \sin 2x \sin y e^{-5t} + 7 \sin 3x \sin 2y e^{-13t}.$$

Here is another example. Suppose that $a = b = \pi$ and

$$h(x, y) = p(x)q(y),$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \leq x \leq \pi/2, \\ \pi - x, & \text{for } \pi/2 \leq x \leq \pi, \end{cases} \qquad q(y) = \begin{cases} y, & \text{for } 0 \leq y \leq \pi/2, \\ \pi - y, & \text{for } \pi/2 \leq y \leq \pi. \end{cases}$$

In this case,

$$\begin{aligned} b_{mn} &= \left(\frac{2}{\pi}\right)^2 \int_0^\pi \left[\int_0^\pi p(x)q(y) \sin mx dx\right] \sin ny dy \\ &= \left(\frac{2}{\pi}\right)^2 \left[\int_0^\pi p(x) \sin mx dx\right] \left[\int_0^\pi q(y) \sin ny dy\right] \\ &= \left(\frac{2}{\pi}\right)^2 \left[\int_0^{\pi/2} x \sin mx dx + \int_{\pi/2}^\pi (\pi - x) \sin mx dx\right] \\ &\qquad \left[\int_0^{\pi/2} y \sin ny dy + \int_{\pi/2}^\pi (\pi - y) \sin ny dy\right]. \end{aligned}$$

The integration can be carried out just like we did in Section 3.3 to yield

$$\begin{aligned} b_{mn} &= \left(\frac{2}{\pi}\right)^2 \left[\frac{2}{m^2} \sin(m\pi/2)\right] \left[\frac{2}{n^2} \sin(n\pi/2)\right] \\ &= \frac{16}{\pi^2} \left[\frac{1}{m^2} \frac{1}{n^2} \sin(m\pi/2) \sin(n\pi/2)\right]. \end{aligned}$$

Thus in this case, we see that

$$h(x, y) = \frac{16}{\pi^2} \sum_{m,n=1}^\infty \left[\frac{1}{m^2} \frac{1}{n^2} \sin(m\pi/2) \sin(n\pi/2)\right] \sin mx \sin my,$$

and hence

$$u(x, y, t) = \frac{16}{\pi^2} \sum_{m,n=1}^\infty \left[\frac{1}{m^2} \frac{1}{n^2} \sin(m\pi/2) \sin(n\pi/2)\right] \sin mx \sin my e^{-(m^2+n^2)t}.$$

Exercises:

5.3.1. Solve the following initial value problem for the heat equation in a square region: Find $u(x,y,t)$, where $0 \le x \le \pi, 0 \le y \le \pi$ and $t \ge 0$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = 2\sin x \sin y + 5\sin 2x \sin y.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \lambda f(x,y), \quad f(x,0) = f(x,\pi) = f(0,y) = f(\pi,y) = 0,$$

are of the form

$$\lambda = -m^2 - n^2, \qquad f(x,y) = b_{mn} \sin mx \sin ny,$$

for $m = 1,2,3,\dots$ and $n = 1,2,3,\dots$, where b_{mn} is a constant.

5.3.2. Solve the following initial value problem for the heat equation in a square region: Find $u(x,y,t)$, where $0 \le x \le \pi, 0 \le y \le \pi$ and $t \ge 0$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = 2(\sin x)y(\pi - y).$$

5.3.3. Solve the following initial value problem in a square region: Find $u(x,y,t)$, where $0 \le x \le \pi, 0 \le y \le \pi$ and $t \ge 0$, such that

$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = 2\sin x \sin y + 3\sin 2x \sin y.$$

5.3.4. Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$u(x,0,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,\pi,t) = \sin x - 2\sin 2x + 3\sin 3x.$$

5.4 Two derivations of the wave equation

We have already seen how the one-dimensional wave equation describes the motion of a vibrating string. In this section we will show that the motion of a vibrating membrane is described by the two-dimensional wave equation, while sound waves are described by a three-dimensional wave equation. In fact, we will see that sound waves arise from "linearization" of the nonlinear equations of fluid mechanics.

The vibrating membrane. Suppose that a homogeneous membrane is fastened down along the boundary of a region D in the (x, y)-plane. Suppose, moreover, that a point on the membrane can move only in the vertical direction, and let $u(x, y, t)$ denote the height of the point with coordinates (x, y) at time t.

If ρ denotes the density of the membrane (assumed to be constant), then by Newton's second law, the force $\mathbf{F}$ acting on a small rectangular piece of the membrane located at (x, y) with sides of length dx and dy is given by the expression

$$\mathbf{F} = \rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy \mathbf{k}.$$

Suppose the force displaces the membrane from a given position $u(x, y)$ to a new position

$$u(x, y) + \eta(x, y),$$

where $\eta(x, y)$ and its derivatives are very small. Then the total work performed by the force $\mathbf{F}$ will be

$$\mathbf{F} \cdot \eta(x, y)\mathbf{k} = \eta(x, y)\rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy.$$

Integrating over the membrane yields an expression for the total work performed when the membrane moves through the displacement η:

$$\text{Work} = \int\int_D \eta(x, y)\rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy. \tag{5.14}$$

On the other hand, the potential energy stored in the membrane is proportional to the extent to which the membrane is stretched. Just as in the case of the vibrating string, this stretching is approximated by the integral

$$\text{Potential energy} = \frac{T}{2} \int\int_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy,$$

where T is a constant, called the *tension* in the membrane. Replacing u by $u+\eta$ in this integral yields

$$\text{New potential energy} = \frac{T}{2} \int\int_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy$$

$$+T \int \int_D \left[\left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) + \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) \right] dxdy$$

$$+\frac{T}{2} \int \int_D \left[\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right] dxdy.$$

If we neglect the last term in this expression (which is justified if η and its derivatives are assumed to be small), we find that

$$\text{New potential energy} - \text{Old potential energy} = \int \int_D T \nabla u \cdot \nabla \eta dxdy.$$

It follows from the divergence theorem in the plane and the fact that η vanishes on the boundary ∂D that

$$\int \int_D T \nabla u \cdot \nabla \eta dxdy + \int \int_D T \eta \nabla \cdot \nabla u dxdy = \int_{\partial D} T \eta \nabla u \cdot \mathbf{N} ds = 0,$$

and hence

$$\text{Change in potential} = - \int \int_D \eta(x, y) T (\nabla \cdot \nabla u)(x, y) dxdy. \tag{5.15}$$

The work performed must be minus the change in potential energy, so it follows from (5.14) and (5.15) that

$$\int \int_D \eta(x, y) \rho \frac{\partial^2 u}{\partial t^2}(x, y) dxdy = \int \int_D \eta(x, y) T (\nabla \cdot \nabla u)(x, y) dxdy.$$

Since this equation holds for all choices of η, it follows that

$$\rho \frac{\partial^2 u}{\partial t^2} = T \nabla \cdot \nabla u,$$

which simplifies to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla \cdot \nabla u, \quad \text{where} \quad c^2 = \frac{T}{\rho},$$

or equivalently

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

This is just the wave equation.

The equations of a perfect gas and sound waves. Next, we will describe *Euler's equations* for a perfect gas in (x, y, z)-space.[1] Euler's equations are expressed in terms of the quantities,

$$\mathbf{v}(x, y, z, t) = (\text{velocity of the gas at } (x, y, z) \text{ at time } t),$$

[1] For a complete derivation of these equations, see Chapter 9 of Alexander Fetter and John Walecka, *Theoretical mechanics of particles and continua*, McGraw-Hill, New York, 1980, or Chapter 1 of Alexandre Chorin and Jerrold Marsden, *A mathematical introduction to fluid mechanics*, third edition, Springer, 1993.

$$\rho(x, y, z, t) = (\text{density at } (x, y, z) \text{ at time } t),$$
$$p(x, y, z, t) = (\text{pressure at } (x, y, z) \text{ at time } t).$$

The first of the Euler equations is the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5.16}$$

To derive this equation, we represent the fluid flow by the vector field

$$\mathbf{F} = \rho \mathbf{v},$$

so that the surface integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA$$

represents the rate at which the fluid is flowing across $\mathbf{S}$ in the direction of $\mathbf{N}$. We assume that no fluid is being created or destroyed. Then the rate of change of the mass of fluid within D is given by two expressions,

$$\int\int\int_D \frac{\partial \rho}{\partial t}(x, y, z, t) dxdydz \qquad \text{and} \qquad -\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA,$$

which must be equal. It follows from the divergence theorem that the second of these expressions is

$$-\int\int\int_D \nabla \cdot \mathbf{F}(x, y, z, t) dxdydz,$$

and hence

$$\int\int\int_D \frac{\partial \rho}{\partial t} dxdydz = -\int\int\int_D \nabla \cdot \mathbf{F} dxdydz.$$

Since this equation must hold for *every* region D in (x, y, z)-space, we conclude that the integrands must be equal,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{F} = -\nabla \cdot (\rho \mathbf{v}),$$

which is just the equation of continuity.

The second of the Euler equations is simply Newton's second law of motion,

$$(\text{mass density})(\text{acceleration}) = (\text{force density}).$$

We make an assumption that the only force acting on a fluid element is due to the pressure, an assumption which is not unreasonable in the case of a perfect gas. In this case, it turns out that the pressure is defined in such a way that the force acting on a fluid element is minus the gradient of pressure:

$$\text{Force} = -\nabla p(x(t), y(t), z(t), t). \tag{5.17}$$

The familiar formula Force = Mass $\times$ Acceleration then yields

$$\rho \frac{d}{dt} \left(\mathbf{v}(x(t), y(t), z(t), t) \right) = -\nabla p(x(t), y(t), z(t), t).$$

Using the chain rule,

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}\frac{dt}{dt},$$

we can rewrite this equation as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p. \tag{5.18}$$

Note that this equation is nonlinear because of the term $(\mathbf{v} \cdot \nabla)\mathbf{v}$.

To finish the Euler equations, we need an *equation of state*, which relates pressure and density. The equation of state could be determined by experiment, the simplest equation of state being

$$p = a^2 \rho^\gamma, \tag{5.19}$$

where a^2 and γ are constants. (An ideal monatomic gas has this equation of state with $\gamma = 5/3$.)

The Euler equations (5.16), (5.18), and (5.19) are nonlinear, and hence quite difficult to solve. However, one explicit solution is the case where the fluid is motionless,

$$\rho = \rho_0, \quad p = p_0, \quad \mathbf{v} = 0,$$

where ρ_0 and p_0 satisfy

$$p_0 = a^2 \rho_0^\gamma.$$

Linearizing Euler's equations near this explicit solution gives rise to the linear differential equation which governs propagation of sound waves.

Let us write

$$\rho = \rho_0 + \rho', p = p_0 + p', \mathbf{v} = \mathbf{v}',$$

where ρ', p' and $\mathbf{v}'$ are so small that their squares can be ignored.

Substitution into Euler's equations yields

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot (\mathbf{v}') = 0,$$

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{1}{\rho_0}\nabla p',$$

and

$$p' = [a^2 \gamma (\rho_0)^{(\gamma-1)}]\rho' = c^2 \rho',$$

where c^2 is a new constant. It follows from these three equations that

$$\frac{\partial^2 \rho'}{\partial t^2} = -\rho_0 \left(\nabla \cdot \frac{\partial \mathbf{v}'}{\partial t} \right) = \nabla \cdot \nabla p' = c^2 \nabla \cdot \nabla \rho'.$$

Thus ρ' must satisfy the three-dimensional wave equation

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \nabla \cdot \nabla \rho' = c^2 \left(\frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} + \frac{\partial^2 \rho'}{\partial z^2} \right). \tag{5.20}$$

If the sound wave ρ' is independent of z, (5.20) reduces to

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \left(\frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} \right),$$

exactly the same equation that we obtained for the vibrating membrane.

Remark: The notion of linearization is extremely powerful because it enables us to derive information on the behavior of solutions to the nonlinear Euler equations, which are extremely difficult to solve except for under very special circumstances.

The Euler equations for a perfect gas and the closely related Navier-Stokes equations for an incompressible fluid such as water form basic models for fluid mechanics. In the case of incompressible fluids, the density is constant, so no equation of state is assumed. To allow for viscosity, one adds an additional term to the expression (5.17) for the force acting on a fluid element:

$$\text{Force} = \nu(\Delta \mathbf{v})(x, y, z, t) - \nabla p(x(t), y(t), z(t), t).$$

Here the Laplace operator is applied componentwise and ν is a constant, called the viscosity of the fluid. The equations used by Navier and Stokes to model an incompressible viscous fluid (with ρ constant) are then

$$\nabla \cdot \mathbf{v} = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \Delta \mathbf{v} - \frac{1}{\rho} \nabla p.$$

It is remarkable that these equations, so easily expressed, are so difficult to solve. Indeed, the Navier-Stokes equations form the basis for one of the seven Millenium Prize Problems, singled out by the Clay Mathematics Institute as central problems for mathematics at the turn of the century. If you can show that under reasonable initial conditions, the Navier-Stokes equations possess a unique well-behaved solution, you may be able to win one million dollars. To find more details on the prize offered for a solution, you can consult the web address: http://www.claymath.org/millennium/

Exercise:

5.4.1. Show that if the tension and density of a membrane are given by variable functions $T(x, y)$ and $\rho(x, y)$ respectively, then the motion of the string is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho(x, y)} \nabla \cdot (T(x, y) \nabla u).$$

5.5 Initial value problems for wave equations

The most natural initial value problem for the wave equation is the following:

Let D be a bounded region in the (x, y)-plane which is bounded by a piecewise smooth curve ∂D, and let $h_1, h_2 : D \to \mathsf{R}$ be continuous functions. Find a function $u(x, y, t)$ such that

1. u satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{5.21}$$

2. u satisfies the "Dirichlet boundary condition" $u(x, y, t) = 0$, for $(x, y) \in \partial D$.

3. u satisfies the initial condition

$$u(x, y, 0) = h_1(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = h_2(x, y).$$

Solution of this initial value problem via separation of variables is very similar to the solution of the initial value problem for the heat equation which was presented in Section 5.3.

As before, let us suppose that $D = \{(x, y) \in \mathsf{R}^2 : 0 \le x \le a, 0 \le y \le b\}$, so that the Dirichlet boundary condition becomes

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0.$$

We write

$$u(x, y, t) = f(x, y)g(t),$$

and substitute into (5.21) to obtain

$$f(x, y)g''(t) = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) g(t).$$

Then we divide by $c^2 f(x, y)g(t)$,

$$\frac{1}{c^2 g(t)} g''(t) = \frac{1}{f(x, y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

Once again, we conclude that both sides must be constant. If we let λ denote the constant, we obtain

$$\frac{1}{c^2 g(t)} g''(t) = \lambda = \frac{1}{f(x, y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

This separates into an ordinary differential equation

$$g''(t) = c^2 \lambda g(t), \tag{5.22}$$

and the Helmholtz equation

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \lambda f. \tag{5.23}$$

The Dirichlet boundary condition becomes

$$f(0, y) = f(a, y) = f(x, 0) = f(x, b) = 0.$$

The Helmholtz equation is solved in exactly the same way as before, the only nontrivial solutions being

$$f_{mn}(x, y) = \sin(m\pi x/a)\sin(n\pi y/b), \quad \text{with} \quad \lambda_{mn} = -(m\pi/a)^2 - (n\pi/b)^2.$$

The corresponding solution to (5.22) is

$$g(t) = A\cos(\omega_{mn}t) + B\sin(\omega_{mn}t), \qquad \text{where } \omega_{mn} = c\sqrt{-\lambda_{mn}}.$$

Thus we obtain a product solution to the wave equation with Dirichlet boundary conditions:

$$u(x, y, t) = \sin(m\pi x/a)\sin(n\pi y/b)[A\cos(\omega_{mn}t) + B\sin(\omega_{mn}t)].$$

The general solution to to the wave equation with Dirichlet boundary conditions is a superposition of these product solutions,

$$u(x, y, t) = \sum_{m,n=1}^{\infty} \sin(m\pi x/a)\sin(n\pi y/b)[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)].$$

The constants A_{mn} and B_{mn} are determined from the initial conditions.

The initial value problem considered in this section could represent the motion of a vibrating membrane. Just like in the case of the vibrating string, the motion of the membrane is a superposition of infinitely many modes, the mode corresponding to the pair (m, n) oscillating with frequency $\omega_{mn}/2\pi$. The lowest frequency of vibration or *fundamental frequency* is

$$\frac{\omega_{11}}{2\pi} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \frac{1}{2}\sqrt{\frac{T}{\rho}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]}.$$

Exercises:

5.5.1. Solve the following initial value problem for a vibrating square membrane: Find $u(x, y, t), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0,$$

$$u(x,y,0) = 3\sin x \sin y + 7\sin 2x \sin y, \quad \frac{\partial u}{\partial t}(x,y,0) = 0.$$

5.5.2. Solve the following initial value problem for a vibrating square membrane: Find $u(x,y,t), 0 \le x \le \pi, 0 \le y \le \pi$, such that

$$\frac{\partial^2 u}{\partial t^2} = 4\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = 0, \quad \frac{\partial u}{\partial t}(x,y,0) = 2\sin x \sin y + 13\sin 2x \sin y.$$

5.5.3. Solve the following initial value problem for a vibrating square membrane: Find $u(x,y,t), 0 \le x \le \pi, 0 \le y \le \pi$, such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = p(x)q(y), \quad \frac{\partial u}{\partial t}(x,y,0) = 0,$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi, \end{cases} \qquad q(y) = \begin{cases} y, & \text{for } 0 \le y \le \pi/2, \\ \pi - y, & \text{for } \pi/2 \le y \le \pi. \end{cases}$$

5.6 The Laplace operator in polar coordinates

In order to solve the heat equation over a circular plate, or to solve the wave equation for a vibrating circular drum, we need to express the Laplace operator in polar coordinates (r,θ). These coordinates are related to the standard Euclidean coordinates by the formulae

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The tool we need to express the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in terms of polar coordinates is just the chain rule, which we studied earlier in the course. Although straigtforward, the calculation is somewhat lengthy. Since

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta, \quad \frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta,$$

it follows immediately from the chain rule that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y}.$$

Similarly, since

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta,$$

it follows that

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y}.$$

We can write the results as operator equations,

$$\frac{\partial}{\partial r} = (\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \theta} = (-r\sin\theta)\frac{\partial}{\partial x} + (r\cos\theta)\frac{\partial}{\partial y}.$$

For the second derivatives, we find that

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}(\frac{\partial u}{\partial r}) = \frac{\partial}{\partial r}\left[\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right] = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}.$$

Similarly,

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}(\frac{\partial u}{\partial \theta}) = \frac{\partial}{\partial \theta}\left[-r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\right]$$

$$= -r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) - r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y}$$

$$= r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} - r\frac{\partial u}{\partial r},$$

which yields

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}.$$

Adding these results together, we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r},$$

or equivalently,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Finally, we can write this result in the form

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{5.24}$$

This formula for the Laplace operator, together with the theory of Fourier series, allows us to solve the Dirichlet problem for Laplace's equation in a disk. Indeed, we can now formulate the Dirichlet problem as follows: Find $u(r, \theta)$, for $0 < r \leq 1$ and $\theta \in mathbbR$, such that

1. u satisfies Laplace's equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \tag{5.25}$$

2. u satisfies the periodicity condition $u(r, \theta + 2\pi) = u(r, \theta)$,
3. u is well-behaved near $r = 0$,
4. u satisfies the boundary condition $u(1, \theta) = h(\theta)$, where $h(\theta)$ is a given well-behaved function satisfying the periodicity condition $h(\theta + 2\pi) = h(\theta)$.

The first three of these conditions are homogeneous linear. To treat these conditions via the method of separation of variables, we set

$$u(r, \theta) = R(r)\Theta(\theta), \quad \text{where} \quad \Theta(\theta + 2\pi) = \Theta(\theta).$$

Substitution into (5.25) yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0.$$

We multiply through by r^2,

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + R\frac{d^2\Theta}{d\theta^2} = 0,$$

and divide by $-R\Theta$ to obtain

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}.$$

The left-hand side of this equation does not depend on θ while the right-hand side does not depend on r. Thus neither side can depend on either θ or r, and hence both sides must be constant:

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda.$$

Thus the partial differential equation divides into two ordinary differential equations

$$\frac{d^2\Theta}{d\theta^2} = \lambda\Theta, \quad \Theta(\theta + 2\pi) = \Theta(\theta),$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda R.$$

We have seen the first of these equations before when we studied heat flow in a circular wire, and we recognize that with the periodic boundary conditions, the only nontrivial solutions are

$$\lambda = 0, \quad \Theta = \frac{a_0}{2},$$

where a_0 is a constant, and

$$\lambda = -n^2, \quad \Theta = a_n \cos n\theta + b_n \sin n\theta,$$

where a_n and b_n are constants, for n a positive integer. Substitution into the second equation yields

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 R = 0.$$

If $n = 0$, the equation for R becomes

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0,$$

which is easily solved to yield

$$R(r) = A + B \log r,$$

where A and B are constants of integration. In order for this equation to be well-behaved as $r \to 0$ we must have $B = 0$, and the solution $u_0(r, \theta)$ to Laplace's equation in this case is constant.

When $n \neq 0$, the equation for R is a Cauchy-Euler equidimensional equation and we can find a nontrivial solution by setting

$$R(r) = r^m.$$

Then the equation becomes

$$r\frac{d}{dr}\left(r\frac{d}{dr}(r^m)\right) = n^2 r^m,$$

and carrying out the differentiation on the left-hand side yields the characteristic equation

$$m^2 - n^2 = 0,$$

which has the solutions $m = \pm n$. In this case, the solution is

$$R(r) = Ar^n + Br^{-n}.$$

Once again, in order for this equation to be well-behaved as $r \to 0$ we must have $B = 0$, so $R(r)$ is a constant multiple of r^n, and

$$u_n(r, \theta) = a_n r^n \cos n\theta + b_n r^n \sin n\theta.$$

The general solution to (5.25) which is well-behaved at $r = 0$ and satisfies the periodicity condition $u(r, \theta + 2\pi) = u(r, \theta)$ is therefore

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

where $a_0, a_1, \dots, b_1, \dots$ are constants. To determine these constants we must apply the boundary condition:

$$h(\theta) = u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

We conclude that the constants $a_0, a_1, \dots, b_1, \dots$ are simply the Fourier coefficients of h.

Exercises:

5.6.1. Solve the following boundary value problem for Laplace's equation in a disk: Find $u(r, \theta), 0 < r \leq 1$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad \text{u well-behaved near } r = 0$$

and

$$u(1, \theta) = h(\theta), \quad \text{where} \quad h(\theta) = 1 + \cos\theta - 2\sin\theta + 4\cos 2\theta.$$

5.6.2. Solve the following boundary value problem for Laplace's equation in a disk: Find $u(r, \theta), 0 < r \leq 1$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad \text{u well-behaved near } r = 0$$

and

$$u(1, \theta) = h(\theta),$$

where $h(\theta)$ is the periodic function such that

$$h(\theta) = |\theta|, \qquad \text{for } -\pi \leq \theta \leq \pi.$$

5.6.3. Solve the following boundary value problem for Laplace's equation in an annular region: Find $u(r, \theta), 1 \leq r \leq 2$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad u(1, \theta) = 1 + 3\cos\theta - \sin\theta + \cos 2\theta$$

and

$$u(2, \theta) = 2\cos\theta + 4\cos 2\theta.$$

5.7 Eigenvalues of the Laplace operator

We would now like to consider the heat equation for a room whose shape is given by a well-behaved but otherwise arbitrary bounded region D in the (x, y)-plane, the boundary ∂D being a well-behaved curve. We would also like to consider the wave equation for a vibrating drum in the shape of such a region D. Both cases quickly lead to the eigenvalue problem for the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let's start with the heat equation

$$\frac{\partial u}{\partial t} = c^2 \Delta u, \tag{5.26}$$

with the Dirichlet boundary condition,

$$u(x, y, t) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

To solve this equation, we apply separation of variables as before, setting

$$u(x, y, t) = f(x, y)g(t),$$

and substitute into (5.26) to obtain

$$f(x, y)g'(t) = c^2(\Delta f)(x, y)g(t).$$

Then we divide by $c^2 f(x, y)g(t)$,

$$\frac{1}{c^2 g(t)} g'(t) = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

The left-hand side of this equation does not depend on x or y while the right-hand side does not depend on t. Hence both sides equal a constant λ, and we obtain

$$\frac{1}{c^2 g(t)} g'(t) = \lambda = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

This separates into

$$g'(t) = c^2 \lambda g(t) \qquad \text{and} \qquad (\Delta f)(x, y) = \lambda f(x, y)$$

in which f is subject to the boundary condition,

$$f(x, y) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

The same method can be used to treat the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \tag{5.27}$$

with the Dirichlet boundary condition,

$$u(x, y, t) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

This time, substitution of $u(x, y, t) = f(x, y)g(t)$ into (5.27) yields

$$f(x, y)g''(t) = c^2(\Delta f)(x, y)g(t), \qquad \text{or} \qquad \frac{1}{c^2 g(t)}g''(t) = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

Once again, both sides must equal a constant λ, and we obtain

$$\frac{1}{c^2 g(t)}g''(t) = \lambda = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

This separates into

$$g''(t) = \lambda g(t) \qquad \text{and} \qquad (\Delta f)(x, y) = \lambda f(x, y),$$

in which f is once again subject to the boundary condition,

$$f(x, y) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

In both cases, we must solve the "eigenvalue problem" for the Laplace operator with Dirichlet boundary conditions. If λ is a real number, let

$$W_\lambda = \{\text{smooth functions } f(x, y) : \Delta f = \lambda f, f|\partial D = 0\}.$$

We say that λ is an *eigenvalue* of the Laplace operator Δ on D if $W_\lambda \neq 0$. Nonzero elements of W_λ are called *eigenfunctions* and W_λ itself is called the *eigenspace* for eigenvalue λ. The dimension of W_λ is called the *multiplicity* of the eigenvalue λ. The *eigenvalue problem* consist of finding the eigenvalues λ, and a basis for each nonzero eigenspace.

Once we have solved the eigenvalue problem for a given region D in the (x, y)-plane, it is easy to solve the initial value problem for the heat equation or the wave equation on this region. To do so requires only that we substitute the values of λ into the equations for g. In the case of the heat equation,

$$g'(t) = c^2\lambda g(t) \Rightarrow g(t) = (\text{constant})e^{c^2\lambda t},$$

while in the case of the wave equation,

$$g''(t) = c^2\lambda g(t) \Rightarrow g(t) = (\text{constant})\sin(c\sqrt{-\lambda}(t - t_0)).$$

In the second case, the eigenvalues determine the frequencies of a vibrating drum which has the shape of D.

Theorem. *All of the eigenvalues of Δ are negative, and each eigenvalue has finite multiplicity. The eigenvalues can be arranged in a sequence*

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots,$$

with $\lambda_n \to -\infty$. Every well-behaved function can be represented as a convergent sum of eigenfunctions.[2]

Although the theorem is reassuring, it is usually quite difficult to determine the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ explicitly for a given region in the plane.

There are two important cases in which the eigenvalues can be explicitly determined. The first is the case where D is the rectangle,

$$D = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}.$$

We saw how to solve the eigenvalue problem

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f, \qquad f(x, y) = 0 \quad \text{for} \quad (x, y) \in \partial D,$$

when we discussed the heat and wave equations for a rectangular region. The nontrivial solutions are

$$\lambda_{mn} = -\left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi n}{b}\right)^2, \quad f_{mn}(x, y) = \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right),$$

where m and n are positive integers.

A second case in which the eigenvalue problem can be solved explicitly is that of the disk,

$$D = \{(x, y) : x^2 + y^2 \le a^2\},$$

where a is a positive number, as we will see in the next section.[3]

Exercises:

5.7.1. Show that among all rectangular vibrating membranes of area one, the square has the lowest fundamental frequency of vibration by minimizing the function

$$f(a, b) = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

subject to the constraints $ab = 1$, $a, b > 0$. Hint: One can eliminate b by setting $a = 1/b$ and then find the minimum of the function

$$g(a) = \left(\frac{\pi}{a}\right)^2 + (\pi a)^2,$$

when $a > 0$.

5.7.2. Let D be a finite region in the (x, y)-plane bounded by a smooth curve ∂D. Suppose that the eigenvalues for the Laplace operator Δ with Dirichlet boundary conditions on D are λ_1, λ_2, ..., where

$$0 > \lambda_1 > \lambda_2 > \cdots,$$

[2]Note the similarity between the statement of this theorem and the statement of the theorem presented in Section 4.7. In fact, the techniques used to prove the two theorems are also quite similar.

[3]A few more cases are presented in advanced texts, such as Courant and Hilbert, *Methods of mathematical physics I*, New York, Interscience, 1953. See Chapter V, §16.3.

each eigenvalue having multiplicity one. Suppose that $\phi_n(x,y)$ is a nonzero eigenfunction for eigenvalue λ_n. Show that the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

with Dirichlet boundary conditions (u vanishes on ∂D) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \phi_n(x,y) e^{\lambda_n t},$$

where the b_n's are arbitrary constants.

5.7.3. Let D be a finite region in the (x,y)-plane as in the preceding problem. Suppose that the eigenvalues for the Laplace operator Δ on D are λ_1, λ_2, ..., once again. Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u,$$

together with Dirichlet boundary conditions (u vanishes on ∂D) and the initial condition

$$\frac{\partial u}{\partial t}(x,y,0) = 0,$$

is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \phi_n(x,y) \cos(\sqrt{-\lambda_n} t),$$

where the b_n's are arbitrary constants.

5.8 Eigenvalues of the disk

To calculate the eigenvalues of the disk, it is convenient to utilize polar coordinates r, θ in terms of which the Laplace operator is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \lambda f, \quad f|\partial D = 0. \tag{5.28}$$

Once again, we use separation of variables and look for product solutions of the form

$$f(r,\theta) = R(r)\Theta(\theta), \quad \text{where} \quad R(a) = 0, \quad \Theta(\theta + 2\pi) = \Theta(\theta).$$

Substitution into (5.28) yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = \lambda R\Theta.$$

We multiply through by r^2,

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + R\frac{d^2\Theta}{d\theta^2} = \lambda r^2 R\Theta,$$

and divide by $R\Theta$ to obtain

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}.$$

Note that in this last equation, the left-hand side does not depend on θ while the right-hand side does not depend on r. Thus neither side can depend on either θ or r, and hence both sides must be constant:

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu.$$

Thus in the manner now familiar, the partial differential equation divides into two ordinary differential equations

$$\frac{d^2\Theta}{d\theta^2} = \mu\Theta, \quad \Theta(\theta + 2\pi) = \Theta(\theta),$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda r^2 R = -\mu R, \quad R(a) = 0.$$

Once again, the only nontrivial solutions are

$$\mu = 0, \quad \Theta = \frac{a_0}{2}$$

and

$$\mu = -n^2, \quad \Theta = a_n \cos n\theta + b_n \sin n\theta,$$

for n a positive integer. Substitution into the second equation yields

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (-\lambda r^2 - n^2)R = 0.$$

Let $x = \sqrt{-\lambda}r$ so that $x^2 = -\lambda r^2$. Then since

$$r\frac{d}{dr} = x\frac{d}{dx},$$

our differential equation becomes

$$x\frac{d}{dx}\left(x\frac{dR}{dx}\right) + (x^2 - n^2)R = 0, \tag{5.29}$$

where R vanishes when $x = \sqrt{-\lambda}a$. If we replace R by y, this becomes

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - n^2)y = 0, \qquad y(\sqrt{-\lambda}a) = 0. \tag{5.30}$$

The differential equation appearing in (5.30) is Bessel's equation, which we studied in Section 1.4.

Recall that in Section 1.4, we found that for each choice of n, Bessel's equation has a one-dimensional space of well-behaved solutions, which are constant multiples of the Bessel function of the first kind $J_n(x)$. Here is an important fact regarding these Bessel functions:

Theorem. *For each nonnegative integer n, $J_n(x)$ has infinitely many positive zeros.*

Graphs of the functions $J_0(x)$ and $J_1(x)$ suggest that this theorem might well be true, but it takes some effort to prove rigorously. For completeness, we sketch the proof for the case of $J_0(x)$ at the end of the section.

The zeros of the Bessel functions are used to determine the eigenvalues of the Laplace operator on the disk. To see how, note first that the boundary condition,

$$y(\sqrt{-\lambda}a) = 0,$$

requires that $\sqrt{-\lambda}a$ be one of the zeros of $J_n(x)$. Let $\alpha_{n,k}$ denote the k-th positive root of the equation $J_n(x) = 0$. Then

$$\sqrt{-\lambda}a = \alpha_{n,k} \quad \Rightarrow \quad \lambda = -\frac{\alpha_{n,k}^2}{a^2},$$

and

$$R(r) = J_n(\alpha_{n,k} r/a)$$

will be a solution to (5.29) vanishing at $r = a$. Hence, in the case where $n = 0$,

$$\lambda_{0,k} = -\frac{\alpha_{0,k}^2}{a^2},$$

and

$$f_{0,k}(r,\theta) = J_0(\alpha_{0,k} r/a)$$

will be a solution to the eigenvalue problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \lambda f, \quad f|\partial D = 0.$$

Similarly, in the case of general n,

$$\lambda_{n,k} = -\frac{\alpha_{n,k}^2}{a^2},$$

and

$$f_{n,k}(r,\theta) = J_n(\alpha_{n,k} r/a)\cos(n\theta) \quad \text{or} \quad g_{n,k} = J_n(\alpha_{n,k} r/a)\sin(n\theta),$$

for $n = 1, 2, \ldots$, will be solutions to this eigenvalue problem.

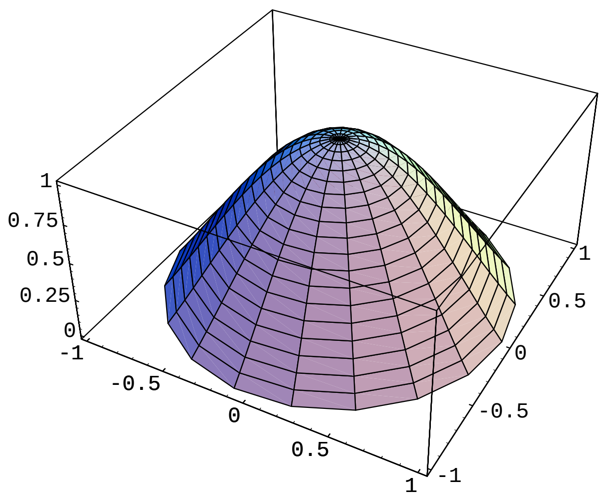

Figure 5.2: Graph of the function $f_{0,1}(r, \theta)$.

Each of these eigenfunctions corresponds to a mode of oscillation for the circular vibrating membrane. Suppose, for example, that $a = 1$. Then

$$\alpha_{0,1} = 2.40483 \Rightarrow \lambda_{0,1} = -5.7832,$$

$$\alpha_{0,2} = 5.52008 \Rightarrow \lambda_{0,2} = -30.471,$$

$$\alpha_{0,3} = 8.65373 \Rightarrow \lambda_{0,3} = -74.887,$$

$$\alpha_{0,4} = 11.7195 \Rightarrow \lambda_{0,3} = -139.039,$$

and so forth. The mode of oscillation corresponding to the function $f_{0,1}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{0,1}}}{2\pi} = \frac{\alpha_{0,1}}{2\pi} = .38274,$$

while the mode of oscillation corresponding to the function $f_{0,2}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{0,2}}}{2\pi} = \frac{\alpha_{0,2}}{2\pi} = .87855.$$

Similarly,

$$\alpha_{1,1} = 3.83171 \Rightarrow \lambda_{1,1} = -14.682,$$

$$\alpha_{2,1} = 5.13562 \Rightarrow \lambda_{2,1} = -26.3746,$$

and hence the mode of oscillation corresponding to the functions $f_{1,1}$ and $g_{1,1}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{1,1}}}{2\pi} = \frac{\alpha_{1,1}}{2\pi} = .60984,$$

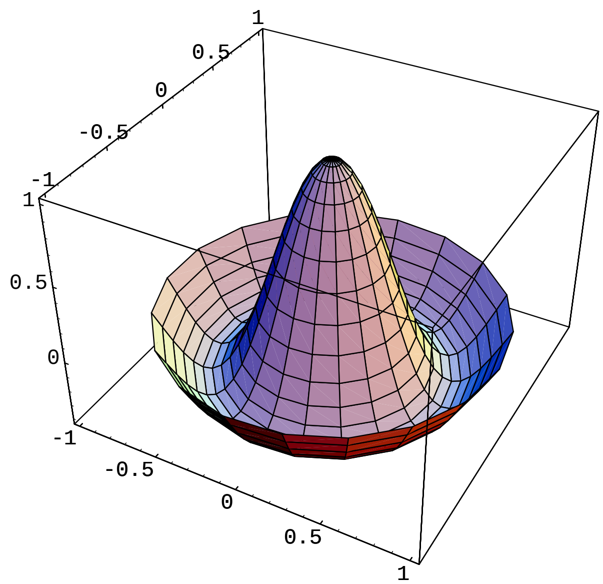

Figure 5.3: Graph of the function $f_{0,2}(r, \theta)$.

while the mode of oscillation corresponding to the functions $f_{2,1}$ and $g_{2,1}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{2,1}}}{2\pi} = \frac{\alpha_{2,1}}{2\pi} = .81736.$$

A general vibration of the vibrating membrane will be a superposition of these modes of oscillation. The fact that the frequencies of oscillation of a circular drum are not integral multiples of a single fundamental frequency (as in the case of the violin string) limits the extent to which a circular drum can be tuned to a specific tone.

Proof that $J_0(x)$ has infinitely many positive zeros: First, we make a change of variables $x = e^z$ and note that as z ranges over the real numbers, the corresponding variable x ranges over all the positive real numbers. Since

$$dx = e^z dz, \qquad \frac{d}{dx} = \frac{1}{e^z}\frac{d}{dz} \qquad \text{and hence} \qquad x\frac{d}{dx} = e^z\frac{1}{e^z}\frac{d}{dz} = \frac{d}{dz}.$$

Thus Bessel's equation (5.30) in the case where $n = 0$ becomes

$$\frac{d^2y}{dz^2} + e^{2z}y = 0. \tag{5.31}$$

Suppose that $z_0 > 1$ and $y(z)$ is a solution to (5.31) with $y(z_0) \neq 0$. We claim that $y(z)$ must change sign at some point between z_0 and $z_0 + \pi$.

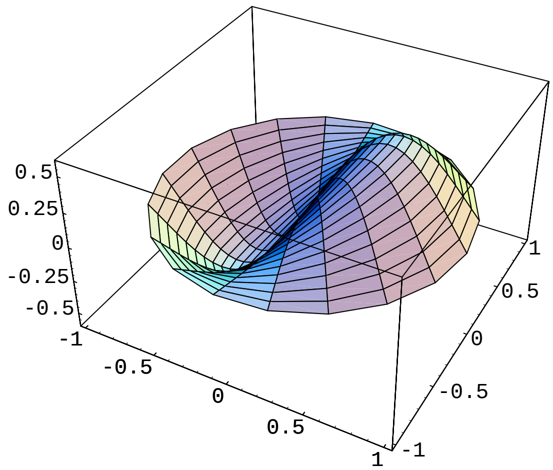

Figure 5.4: Graph of the function $f_{1,1}(r, \theta)$.

Assume first that $y(z_0) > 0$. Let $f(z) = \sin(z - z_0)$. Since $f''(z) = -f(z)$ and $y(z)$ satisfies (5.31),

$$\frac{d}{dz}\left[y(z)f'(z) - y'(z)f(z)\right] = y(z)f''(z) - y''(z)f(z)$$

$$= -y(z)f(z) + e^{2z}y(z)f(z) = (e^{2z} - 1)y(z)f(z).$$

Note that $f(z) > 0$ for z between z_0 and $z_0 + \pi$. If also $y(z) > 0$ for z between z_0 and $z_0 + \pi$, then $y(z)f(z) > 0$ and

$$0 < \int_{z_0}^{z_0+\pi} (e^{2z} - 1)y(z)f(z)dz = \left[y(z)f'(z) - y'(z)f(z)\right]_{z_0}^{z_0+\pi}$$

$$= y(z_0 + \pi)f'(z_0 + \pi) - y(z_0)f(z_0) = -y(z_0 + \pi) - y(z_0) < 0,$$

a contradiction. Hence our assumption that $y(z)$ be postive for z between z_0 and $z_0 + \pi$ must be incorrect, $y(z)$ must change sign at some z between z_0 and $z_0 + \pi$, and hence $y(z)$ must be zero at some point in this interval.

If $y(z_0) < 0$, just apply the same argument to $-y$. In either case, we conclude that $y(z)$ must be zero at some point in any interval to the right of $z = 1$ of length at least π. It follows that the solution to (5.31) must have infinitely many zeros in the region $z > 1$, and $J_0(x)$ must have infinitely many positive zeros, as claimed.

The fact that $J_n(x)$ has infinitely many positive zeros could be proven in a similar fashion.

Exercises:

5.8.1. Solve the following initial value problem for the heat equation in a disk: Find $u(r,\theta,t), 0 < r \leq 1$, such that

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r,\theta+2\pi,t) = u(r,\theta,t), \quad \text{u well-behaved near } r=0,$$

$$u(1,\theta,t) = 0,$$

$$u(r,\theta,0) = J_0(\alpha_{0,1}r) + 3J_0(\alpha_{0,2}r) - 2J_1(\alpha_{1,1}r)\sin\theta + 4J_2(\alpha_{2,1}r)\cos 2\theta.$$

5.8.2. Solve the following initial value problem for the vibrating circular membrane: Find $u(r,\theta,t), 0 < r \leq 1$, such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r,\theta+2\pi,t) = u(r,\theta,t), \quad \text{u well-behaved near } r=0,$$

$$u(1,\theta,t) = 0, \quad \frac{\partial u}{\partial t}(r,\theta,0) = 0,$$

$$u(r,\theta,0) = J_0(\alpha_{0,1}r) + 3J_0(\alpha_{0,2}r) - 2J_1(\alpha_{1,1}r)\sin\theta + 4J_2(\alpha_{2,1}r)\cos 2\theta.$$

5.8.3. (For students with access to Mathematica) a. Run the following Mathematica program to sketch the Bessel function $J_0(x)$:

```
n=0; Plot[ BesselJ[n,x], {x,0,15}]
```

b. From the graph it is clear that the first root of the equation $J_0(x) = 0$ is near 2. Run the following Mathematica program to find the first root $\alpha_{0,1}$ of the Bessel function $J_0(x)$:

```
n=0; FindRoot[ BesselJ[n,x] == 0 ,{x,2}]
```

Find the next two nonzero roots of the Bessel function $J_0(x)$.

c. Modify the programs to sketch the Bessel functions $J_1(x), \ldots, J_5(x)$, and determine the first three nonzero roots of each of these Bessel functions.

5.8.4. Which has a lower fundamental frequency of vibration, a square drum or a circular drum of the same area?

5.9 Fourier analysis for the circular vibrating membrane*

To finish up the solution to the initial value problem for arbitrary initial displacements of the vibrating membrane, we need to develop a theory of generalized

Fourier series which gives a decomposition into eigenfunctions which solve the eigenvalue problem.

Such a theory can be developed for an arbitrary bounded region D in the (x, y)-plane bounded by a smooth closed curve ∂D. Let V denote the space of smooth functions $f : D \to \mathsf{R}$ whose restrictions to ∂D vanish. It is important to observe that V is a "vector space": the sum of two elements of V again lies in V and the product of an element of V with a constant again lies in V.

We define an inner product $\langle , \rangle$ on V by

$$\langle f, g\rangle = \int\int_D f(x, y)g(x, y)dxdy.$$

Lemma. *With respect to this inner product, eigenfunctions corresponding to distinct eigenvalues are perpendicular; if f and g are smooth functions vanishing on ∂D such that*

$$\Delta f = \lambda f, \quad \Delta g = \mu g, \tag{5.32}$$

then either $\lambda = \mu$ or $\langle f, g\rangle = 0$.

The proof of this lemma is a nice application of Green's theorem. Indeed, it follows from Green's theorem that

$$\int_{\partial D} -f\frac{\partial g}{\partial y}dx + f\frac{\partial g}{\partial x}dy = \int\int_D \left(\frac{\partial}{\partial x}[f(\partial g/\partial x)] - \frac{\partial}{\partial y}[-f(\partial g/\partial y)]\right) dxdy$$

$$= \int\int_D \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] dxdy + \int\int_D f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) dxdy.$$

Hence if the restriction of f to ∂D vanishes,

$$\int\int_D f\Delta g dxdy = -\int\int_D \nabla f \cdot \nabla g dxdy.$$

Similarly, if the restriction of g to ∂D vanishes,

$$\int\int_D g\Delta f dxdy = -\int\int_D \nabla f \cdot \nabla g dxdy.$$

Thus if f and g lie in V,

$$\langle f, \Delta g\rangle = -\int\int_D \nabla f \cdot \nabla g dxdy = \langle g, \Delta f\rangle.$$

In particular, if (5.32) holds, then

$$\mu\langle f, g\rangle = \langle f, \Delta g\rangle = \langle \Delta f, g\rangle = \lambda\langle f, g\rangle,$$

and hence

$$(\lambda - \mu)\langle f, g\rangle = 0.$$

It follows immediately that either $\lambda - \mu = 0$ or $\langle f, g \rangle = 0$, and the lemma is proven.

Now let us focus on the special case in which D is a circular disk. Recall the problem that we want to solve, in terms of polar coordinates: Find

$$u(r, \theta, t), 0 < r \leq 1,$$

so that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2},$$

$$u(r, \theta + 2\pi, t) = u(r, \theta, t), \quad \text{u well-behaved near } r = 0,$$

$$u(1, \theta, t) = 0,$$

$$u(r, \theta, 0) = h(r, \theta), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

It follows from the argument in the preceding section that the general solution to the homogeneous linear part of the problem can be expressed in terms of the eigenfunctions

$$f_{n,k}(r, \theta) = J_n(\alpha_{n,k} r/a) \cos(n\theta), \quad g_{n,k} = J_n(\alpha_{n,k} r/a) \sin(n\theta).$$

Indeed, the general solution must be a superposition of the products of these eigenfunctions with periodic functions of t of frequencies $\sqrt{-\lambda_{n,k}}/2\pi$. Thus this general solution is of the form

$$u(r, \theta, t) = \sum_{k=1}^{\infty} a_{0,k} g_{0,k}(r, \theta) \cos(\alpha_{0,k} t)$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [a_{n,k} f_{n,k}(r, \theta) + b_{n,k} g_{n,k}(r, \theta)] \cos(\alpha_{n,k} t). \tag{5.33}$$

We need to determine the $a_{n,k}$'s and the $b_{n,k}$'s so that the inhomogeneous initial condition $u(r, \theta, 0) = h(r, \theta)$ is also satisfied. If we set $t = 0$ in (5.33), this condition becomes

$$\sum_{k=1}^{\infty} a_{0,k} f_{0,k} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [a_{n,k} f_{n,k} + b_{n,k} g_{n,k}] = h. \tag{5.34}$$

Thus we need to express an arbitrary initial displacement $h(r, \theta)$ as a superposition of these eigenfunctions.

It is here that the inner product $\langle , \rangle$ on V comes to the rescue. The lemma implies that eigenfunctions corresponding to distinct eigenvalues are perpendicular. This implies, for example, that

$$\langle f_{0,j}, f_{0,k} \rangle = 0, \quad \text{unless } j = k.$$

Similarly, for arbitrary $n \geq 1$,

$$\langle f_{n,j}, f_{n,k} \rangle = \langle f_{n,j}, g_{n,k} \rangle = \langle g_{n,j}, g_{n,k} \rangle = 0, \quad \text{unless } j = k.$$

The lemma does not immediately imply that

$$\langle f_{n,j}, g_{n,j} \rangle = 0.$$

To obtain this relation, recall that in terms of polar coordinates, the inner product $\langle , \rangle$ on V takes the form

$$\langle f, g \rangle = \int_0^1 \left[\int_0^{2\pi} f(r,\theta) g(r,\theta) r d\theta \right] dr.$$

If we perform integration with respect to θ first, we can conclude from the familiar integral formula

$$\int_0^{2\pi} \sin n\theta \cos n\theta d\theta = 0$$

that

$$\langle f_{n,j}, g_{n,j} \rangle = 0,$$

as desired.

Moreover, using the integral formulae

$$\int_0^{2\pi} \cos n\theta \cos m\theta d\theta = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases}$$

$$\int_0^{2\pi} \sin n\theta \sin m\theta d\theta = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases}$$

$$\int_0^{2\pi} \sin n\theta \cos m\theta d\theta = 0,$$

we can check that

$$\langle f_{n,j}, f_{m,k} \rangle = \langle g_{n,j}, g_{m,k} \rangle = \langle f_{n,j}, g_{m,k} \rangle = 0,$$

unless $m = n$. It then follows from the lemma that

$$\langle f_{n,j}, f_{m,k} \rangle = \langle g_{n,j}, g_{m,k} \rangle = \langle f_{n,j}, g_{m,k} \rangle = 0,$$

unless $j = k$ and $m = n$.

From these relations, it is not difficult to construct formulae for the coefficients of the generalized Fourier series of a given function $h(r, \theta)$. Indeed, if we take the inner product of equation (5.34) with $u_{n,k}$, we obtain

$$a_{n,k} \langle f_{n,k}, f_{n,k} \rangle = \langle h, f_{n,k} \rangle,$$

or equivalently

$$a_{n,k} = \frac{\langle h, f_{n,k} \rangle}{\langle f_{n,k}, f_{n,k} \rangle}. \tag{5.35}$$

Similarly,

$$b_{n,k} = \frac{\langle h, g_{n,k} \rangle}{\langle g_{n,k}, g_{n,k} \rangle}. \tag{5.36}$$

Thus we finally see that the solution to our initial value problem is

$$u(r,\theta,t) = \sum_{k=1}^{\infty} a_{0,k} f_{0,k}(r,\theta) \cos(\alpha_{0,k} t)$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [a_{n,k} f_{n,k}(r,\theta) + b_{n,k} g_{n,k}(r,\theta)] \cos(\alpha_{n,k} t),$$

where the coefficients are determined by the integral formulae (5.35) and (5.36).

Exercises:

5.9.1. (For students with access to Mathematica) Suppose that a circular vibrating drum is given the initial displacement from equilibrium described by the function

$$h(r,\theta) = .1(1-r),$$

for $0 \leq r \leq 1$. In order to find the solution to the wave equation with this initial condition, we need to expand $h(r,\theta)$ in terms of eigenfunctions of Δ. Because of axial symmetry, we can write

$$h(r,\theta) = a_{0,1} J_0(\alpha_{0,1} r) + a_{0,2} J_0(\alpha_{0,2} r) + \dots .$$

a. Use the following Mathematica program to determine the coefficient $a_{0,1}$ in this expansion:

```
d = NIntegrate[r (BesselJ[0,2.40483 r])∧2,{r,0,1}];
n = NIntegrate[r (BesselJ[0,2.40483 r]) .1 (1-r),{r,0,1}];
a01 = n/d
```

b. Use a modification of the program to determine the coefficients $a_{0,2}$, $a_{0,3}$, and $a_{0,4}$.

c. Determine the first four terms in the solution to the initial value problem: Find

$$u(r,\theta,t), 0 < r \leq 1,$$

so that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r, \theta + 2\pi, t) = u(r, \theta, t), \quad \text{u well-behaved near } r = 0,$$
$$u(1, \theta, t) = 0,$$
$$u(r, \theta, 0) = .1(1 - r), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

5.9.2. (For students with access to Mathematica) The following program will sketch graphs of the successive positions of a membrane, vibrating with a superposition of several of the lowest modes of vibration:

```
<< Graphics‘ParametricPlot3D‘
a01 = .2; a02 = .1; a03 = .2; a04 = .1; a11 = .3; b11 = .2;
vibmem = Table[
    CylindricalPlot3D[
        a01 BesselJ[0,2.4 r] Cos[2.4 t]
        + a02 BesselJ[0,5.52 r] Cos[5.52 t]
        + a03 BesselJ[0,8.65 r] Cos[8.65 t]
        + a04 BesselJ[0,11.8 r] Cos[11.8 t]
        + a11 BesselJ[1,3.83 r] Cos[u] Cos[3.83 t]
        + b11 BesselJ[1,3.83 r] Sin[u] Cos[3.83 t],
        {r,0,1}, {u,0,2 Pi}, PlotRange -> {-.5,.5}
    ], {t,0,1.,.1}
]
```

a. Execute this program and then animate the sequence of graphics by running “Animate selected graphics,” from the Graphics menu. (Warning! Generating the sequence of graphics takes lots of memory. If you have insufficient memory for the animation, try inserting `PlotPoints -> 8` within the CylindricalPlot statement.)

b. Replace the values of $a01, \ldots$ with the Fourier coefficients obtained in the preceding problem, execute the resulting program to generate a sequence of graphics, and animate the sequence as before.

Appendix A

Using Mathematica to solve differential equations

In solving differential equations, it is sometimes necessary to do calculations which would be prohibitively difficult to do by hand. Fortunately, computers can do the calculations for us, if they are equiped with suitable software, such as Maple, Matlab, or Mathematica. This appendix is intended to give a brief introduction to the use of Mathematica for doing such calculations.

Most computers which are set up to use Mathematica contain an on-line tutorial, “Tour of Mathematica,” which can be used to give a brief hands-on introduction to Mathematica. Using this tour, it may be possible for many students to learn Mathematica without referring to lengthy technical manuals. However, there is a standard reference on the use of Mathematica, which can be used to answer questions if necessary. It is *The Mathematica Book*, by Stephen Wolfram, Fourth edition, Wolfram Media and Cambridge University Press, 1999.

We give here a very brief introduction to the use of Mathematica. After launching Mathematica, you can use it as a “more powerful than usual graphing calculator.” For example, if you type in

```
(11 - 5)/3
```

the computer will perform the subtraction and division, and respond with

```
Out[1] = 2
```

The notation for multiplication is $*$, so if you type in

```
2 * (7 + 4)
```

the computer will respond with

```
Out[2] = 22
```

You can also use a space instead of $*$ to denote multiplication, so if you input

```
2 8
```

the computer will respond with

```
Out[3] = 16
```

The computer can do exact arithmetic with integers or rational numbers. For example, since $\wedge$ is the symbol used for raising to a power, if you type in

```
2∧150
```

the computer will calculate the 150th power of two and give the exact result:

```
Out[4] = 1427247692705959881058285969449495136382746624
```

On the other hand, the symbol `N` tells the computer to use an approximation with a fixed number of digits of precision, so entering

```
N[2∧150]
```

will give an approximation to the exact answer, expressed in scientific notation:

Out[5] = 1.42725 10^{45}

Real numbers which are not rational, such as π, have infinite decimal expansions which cannot be stored within a computer. However, we can approximate a real number by a finite decimal expansion with a given number of digits of precision. For example, since Mathematica uses the name `E` to denote the number e, typing in

```
N[E]
```

will tell Mathematica to give a rational approximation to e to a standard number of digits of precision:

```
Out[6] = 2.71828
```

In principle, the computer can calculate rational approximations to an arbitrary number of digits of precision. Since the number π is represented by Mathematica as `Pi`, typing in

```
N[Pi,40]
```

will tell Mathematica to give a rational approximation to π to 40 digits of precision:

```
Out[7] = 3.1415926535897932384626433832795028841972
```

The computer can represent not only numbers but functions. Within Mathematica, built-in functions are described by names which begin with capital

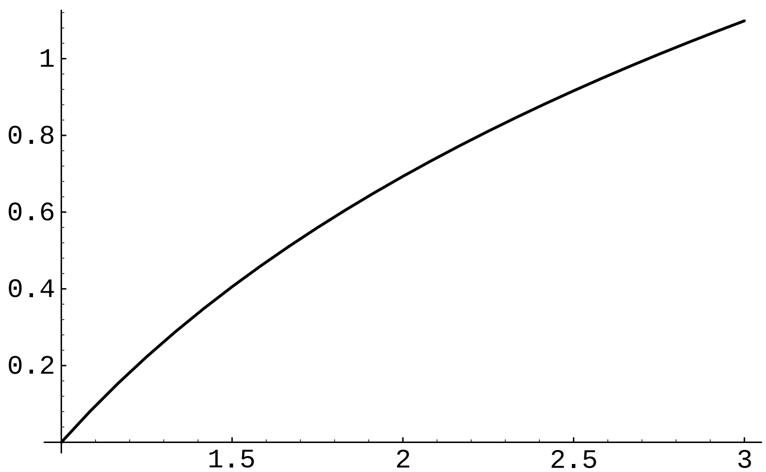

Figure A.1: Graph of the logarithm function.

letters. For example, `Log` denotes the natural base e logarithm function. Thus entering

```
N[Log[2],40]
```

will give the logarithm of 2 with 40 digits of precision:

```
Out[8] = 0.6931471805599453094172321214581765680755
```

One can also plot functions with Mathematica. For example, to plot the logarithm function from the values 1 to 3, one simply inputs

```
Plot[Log[t],{t,1,3}]
```

and Mathematica will automatically produce the plot shown in Figure A.1.

We can also define functions ourselves, being careful not to capitalize them, because capitals are reserved by Mathematica for built-in functions. Thus we can define the function

$$y(t) = ce^{kt}$$

by typing

```
y[t_] := c * E∧(k * t);
```

Mathematica will remember the function we have defined until we quit Mathematica. We must remember the exact syntax of this example (use of the underline character and the colon followed by the equal sign) when defining functions. In this example c and k are *parameters* which can be defined later. Just as in the case of functions, we must be careful to represent parameters by lower case letters, so they will not be confused with built-in constants. Further entry of

```
c = 1; k = 1; N[y[1]]
```

yields the response

```
Out[11] = 2.71828
```

while entering

```
Plot[y[t],{t,0,2}]
```

will give a plot of the function we have defined, for t in the interval $[0, 2]$.

We can use Mathematica to solve matrix differential equations of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{A.1}$$

where A is a square matrix with constant entries.

The first step consists of using Mathematica to find the eigenvalues and eigenvectors of A. To see how this works, we must first become familiar with the way in which Mathematica represents matrices. Since Mathematica reserves upper case letters for descriptions of built-in functions, it is prudent to denote the matrix A by lower case a when writing statements in Mathematica. The matrix

$$A = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$$

can be entered into Mathematica as a collection of row vectors,

```
a = {{-2,5},{1,-3}}
```

with the computer responding by

```
Out[1] = {{-2,5},{1,-3}}
```

Thus a matrix is thought of as a "vector of vectors." Entering

```
MatrixForm[a]
```

will cause the computer to give the matrix in the familiar form

```
Out[2] =  −2  5
          1   −3
```

To find the eigenvalues of the matrix A, we simply type

```
Eigenvalues[a]
```

and the computer will give us the exact eigenvalues

$$\frac{-5 - \sqrt{21}}{2}, \qquad \frac{-5 + \sqrt{21}}{2},$$

which have been obtained by using the quadratic formula. Quite often numerical approximations are sufficient, and these can be obtained by typing

```
eval = Eigenvalues[N[a]]
```

the response this time being

```
Out[4] = {-4.79129, -0.208712}
```

Defining `eval` to be the eigenvalues of A in this fashion, allows us to refer to the eigenvalues of A later by means of the expression `eval`.

We can also find the corresponding eigenvectors for the matrix A by typing

```
evec = Eigenvectors[N[a]]
```

and the computer responds with

```
Out[5] = {{-0.873154, 0.487445}, {0.941409, 0.337267}}
```

Putting this together with the eigenvalues gives the general solution to the original linear system (A.1) for our choice of the matrix A:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -0.873154 \\ 0.487445 \end{pmatrix} e^{-4.79129t} + c_2 \begin{pmatrix} 0.941409 \\ 0.337267 \end{pmatrix} e^{-0.208712t}.$$

Mathematica can also be used to find numerical solutions to nonlinear differential equations. The following Mathematica programs will use Mathematica's differential equation solver (which is called up by the command `NDSolve`), to find a numerical solution to the initial value problem

$$dy/dx = y(1-y), \qquad y(0) = .1,$$

give a table of values for the solution, and graph the solution curve on the interval $0 \leq x \leq 6$. The first step

```
sol := NDSolve[{ y'[x] == y[x] (1 - y[x]), y[0] == .1 },
y, {x,0,6}]
```

generates an "interpolation function" which approximates the solution and calls it sol, an abbreviation for solution. We can construct a table of values for the interpolation function by typing

```
Table[Evaluate[y[x] /. sol], {x,0,6,.1}];
```

or graph the interpolation function by typing

```
Plot[Evaluate[y[x] /. sol], {x,0,6}]
```

This leads to a plot like that shown in Figure A.2.

Readers can modify these simple programs to graph solutions to initial value problems for quite general differential equations of the canonical form

$$\frac{dy}{dx} = f(x,y).$$

All that is needed is to replace the first argument of `NDSolve` with the differential equation one wants to solve, remembering to replace the equal signs with double equal signs, as in the example.

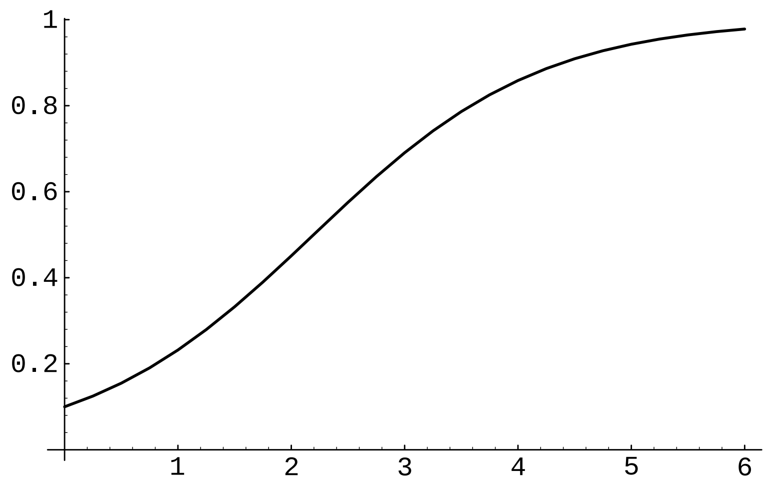

Figure A.2: Solution to $y' = y(1-y)$, $y(0) = .1$.

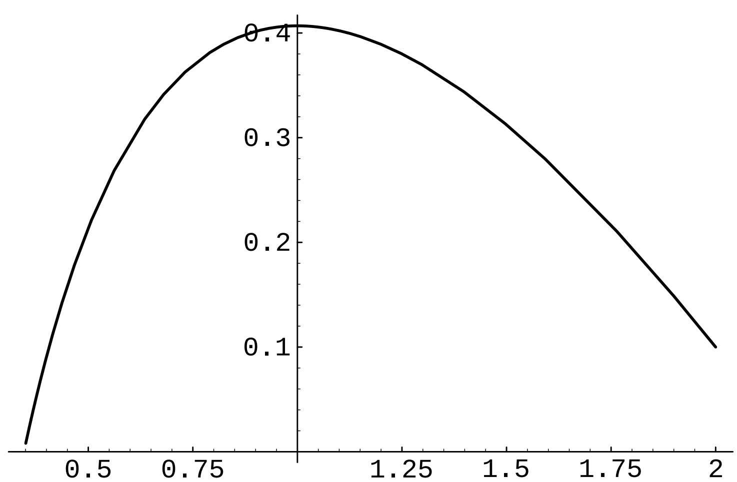

Figure A.3: A parametric plot of a solution to $dx/dt = -xy$, $dy/dt = xy - y$.

In fact, it is no more difficult to treat initial value problems for higher order equations or systems of differential equations. For example, to solve the initial value problem

$$\frac{dx}{dt} = -xy, \quad \frac{dy}{dt} = xy - y, \quad x(0) = 2, \quad y(0) = .1. \tag{A.2}$$

one simply types in

```
sol := NDSolve[{ x'[t] == - x[t] y[t], y'[t] == x[t] y[t] - y[t],
x[0] == 2, y[0] == .1 },
{x,y}, {t,0,10}]
```

Once we have this solution, we can print out a table of values for y by entering

```
Table[Evaluate[y[t] /. sol], {t,0,2,.1}]
```

We can also plot $y(t)$ as a function of t by entering

```
Plot[Evaluate[y[t] /. sol], {t,0,10}]
```

Figure A.3 shows a parametric plot of $(x(t), y(t))$ which is generated by entering

```
ParametricPlot[Evaluate[{ x[t], y[t]} /. sol], {t,0,10}]
```

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