Introduction to Partial Differential Equations

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This chapter introduces basic concepts and definitions for partial differential equations (PDEs) and solutions to a variety of PDEs. Applications of the method of separation of variables are presented for the solution of second-order PDEs. The application of this method involves the use of Fourier series.

Definitions

- Equations involving one or more partial derivatives of a function of two or more independent variables are called _partial differential equations (PDEs)_.

- Well known examples of PDEs are the following equations of mathematical physics in which the notation: $u = \partial u/\partial x$, $u_{xy} = \partial u/\partial y \partial x$, $u_{xx} = \partial^2 u/\partial x^2$, etc., is used:

  [1] One-dimensional wave equation: $u_{tt} = c^2 u_{xx}$

  [2] One-dimensional heat equation: $u_t = c^2 u_{xx}$

  [3] Laplace equation: $u_{xx}+u_{yy} = 0$, (2-D), or $u_{xx}+u_{yy}+u_{zz}= 0$ (3-D)

  [4] Poisson equation: $u_{xx}+u_{yy} = f(x,y)$, (2-D), or $u_{xx}+u_{yy}+u_{zz}= f(x,y,z)$ (3-D)

- The order of the highest derivative is the _order_ of the equation. For example, all of the PDEs in the examples shown above are of the second order.

- A PDE is _linear_ if the dependent variable and its functions are all of first order. All of the PDEs shown above are also linear.

- A PDE is _homogeneous_ if each term in the equation contains either the dependent variable or one of its derivatives. Otherwise, the equation is said to be _non-homogeneous_. Equations [1], [2], and [3] above are homogeneous equations. Equation [4] is non-homogeneous.

- A _solution_ of a PDE in some region R of the space of independent variables is a function, which has all the derivatives that appear on the equation, and satisfies the equation everywhere in R. For example, $u = x^2 - y^2$, $u = e^x \cos(y)$, and $u = \ln(x^2+y^2)$, are all solutions to the two-dimensional Laplace equation (equation [3] above).

- In general there should be as many boundary or initial conditions as the highest order of the corresponding partial derivative. For example, the one dimensional heat equation (equation [2]) applied to a insulated bar of length L, will require an initial condition, say

$$u(x,t=0) = f(x),\ 0 < x < L,$$

as well as two boundary conditions, e.g., $u(x=0,t) = u_0$ and $u(x=L,t) = u_L$, or, $u_x(x=0,t) = u_{x0}$ and $u_x(x=L) = u_{xL}$, or some combination of these, for $t > 0$.

Classification of linear, second-order PDEs

Linear, second-order PDEs, as the examples shown above as equations [1] through [4], are commonly encountered in science and engineering applications. For that reason special attention is paid in this section to this type of equations. First, we learn how to classify linear, second-order PDEs as follows:

An equation of the form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x,y,u,u_x,u_y),$$

is said to be:

- parabolic, if $B^2 - AC = 0$, e.g., heat flow and diffusion-type problems.
- hyperbolic, if $B^2 - AC > 0$, e.g., vibrating systems and wave motion problems.
- elliptic, if $B^2 - AC < 0$, e.g., steady-state, potential-type problems.

Analytical solutions of PDEs

There are a variety of methods for obtaining symbolic, or closed-form, solutions to differential equations. The method of separation of variables can be used to obtain analytical solutions for some simple PDEs. The method consists in writing the general solution as the product of functions of a single variable, then replacing the resulting function into the PDE, and separating the PDE into ODEs of a single variable each. The ODEs are solved separately and their solutions combined into the solution of the PDE.

In many cases, the ODEs resulting from the separation of variables produce solutions that depend on a parameter known as an *eigenvalue* (if the eigenvalue appears in a sine or cosine function that depends on time, it is referred to as an *eigenfrequency*). The solutions involving eigenvalues are known as *eigenfunctions*.

Analytical solutions to parabolic equations: one-dimensional solution of the heat equation

The flow of heat in a thin, laterally insulated homogeneous rod is modeled by

$$\partial u/\partial t = k\cdot(\partial^2 u/\partial x^2),$$

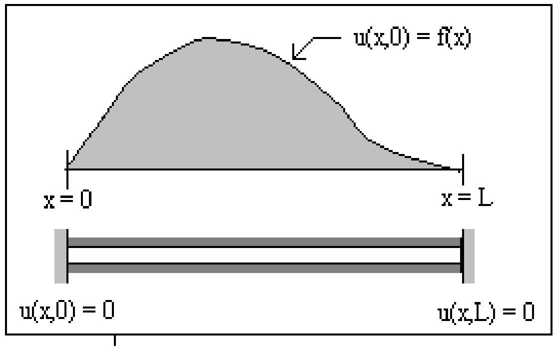
where u = temperature, k = a parameter resulting from combining thermal conductivity and density. The PDE is subject to the initial condition

$$u(x,0) = f(x)$$

and constant-value boundary conditions

$$u(0,t) = u_0, \text{ and } u(L,t) = u_L.$$

The physical phenomenon described by this PDE and its initial and boundary conditions is illustrated in the figure below with $u_0 = u_L = 0$.



We will try to find a solution by the method of separation of variables. This method assumes that the solution, u(x,t), can be expressed as the product of two functions, X(x) and T(t):

$$u(x,t) = X(x)T(t).$$

With this substitution, the initial condition, $u(x,t=0) = f(x) = X(x)T(t)$, can be treated as the set of conditions: $X(x) = f(x)$, when $t = 0$ [i.e., $T(t) = 1$]. Also, the boundary conditions, $u(0,t) = X(0)T(t) = u_0$, and $u(L,t) = X(L)T(t) = u_L$, can be treated as $X(0) = u_0$, and $X(L) = u_L$, as long as $T(t) \neq 0$.

The derivatives of u(x,t) are calculated as follows:

$$\frac{\partial^2}{\partial x^2} u(x,t) \qquad \left(\frac{\partial^2}{\partial x^2} X(x)\right) T(t)$$

and

$$\frac{\partial}{\partial t} u(x,t) \qquad X(x)\left(\frac{\partial}{\partial t} T(t)\right)$$

Replacing these derivatives in the heat equation we get:

$$X(x)\left(\frac{\partial}{\partial t} T(t)\right) = k \left(\frac{\partial^2}{\partial x^2} X(x)\right) T(t)$$

Dividing by $u(x,t) = X(x)T(t)$:

$$\frac{\frac{\partial}{\partial t}\mathrm{T}(t)}{\mathrm{T}(t)} = \frac{k\left(\frac{\partial^2}{\partial x^2}\mathrm{X}(x)\right)}{\mathrm{X}(x)}$$

This result is only possible if both sides of the equation are equal to a constant, say -α, since the left-hand side is only a function of t and the right hand side is only function of x. The left-hand side of the heat equation produces an ODE with independent variable t:

$$\frac{\frac{\partial}{\partial t}\mathrm{T}(t)}{\mathrm{T}(t)} = -\alpha$$

Whose solution is:

$$\mathrm{T}(t) = \mathbf{e}^{(-\alpha\, t)}$$

On the other hand, the right-hand side of the heat equation produces an ODE with independent variable is x:

$$\frac{k\left(\frac{\partial^2}{\partial x^2}\mathrm{X}(x)\right)}{\mathrm{X}(x)} = -\alpha$$

A general solution for *X(x)* is:

$$\mathrm{X}(x) = \_C1\ \sin\left(\sqrt{\frac{\alpha}{k}}\, x\right) + \_C2\ \cos\left(\sqrt{\frac{\alpha}{k}}\, x\right)$$

Next, we replace the boundary condition X(0) = 0, which results in the equation 0 = _C2, or _C2 = 0. With this result, the solution simplifies to:

$$\mathrm{X}(x) = \_C1\ \sin\left(\sqrt{\frac{\alpha}{k}}\, x\right)$$

The second boundary condition, *X(L) = 0*, produces:

$$0 = \_C1 \sin\left(L\sqrt{\frac{\alpha}{k}}\right)$$

The latter result indicates an eigenfunction problem. We need to find all possible values of α for which this equation is satisfied. Since we want $\_C1 \neq 0$, then we set

$$\sin\left(L\sqrt{\frac{\alpha}{k}}\right) = 0$$

This equation has multiple solutions located at,

$$L\sqrt{\frac{\alpha}{k}} = \cdots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$$

i.e.,

$$L\sqrt{\frac{\alpha}{k}} = n\,\pi$$

or,

$$\alpha := \frac{n^2\,\pi^2\,k}{L^2}$$

Therefore, with these values of α the solution for *X(x)* now becomes:

$$\mathrm{X}(x) = \_C1 \sin\left(\sqrt{\frac{n^2\,\pi^2}{L^2}}\,x\right)$$

The value of _C1 remains somewhat arbitrary, requiring a different approach to find it. To simplify notation we will replace _C1 with b_n:

$$X(x) = b_n \sin\left(\sqrt{\frac{n^2\pi^2}{L^2}}\,x\right)$$

With the value of α found earlier, the solution for *T(t)* is now:

$$T(t) = e^{\left(-\frac{n^2\pi^2 k\,t}{L^2}\right)}$$

There will be a different expression for *u(x,t)* = *X(t)T(t)* for each value of n = 0 , 1, 2, 3, ... Therefore, we will call the solution corresponding to a particular value of n $u_n(x,t)$ and write:

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(-\frac{n^2\pi^2 k\,t}{L^2}\right)}$$

The form of the *n*-th solution, u_n, suggests an expansion similar to a Fourier series expansion for the overall solution, *u(x,t)*, restricting the values of n to positive integers, i.e.,

$$u_n(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right).$$

with the values of b_n obtained from the boundary condition, *u(x,0) = f(x)*, i.e.,

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) b_n = f(x)$$

The latter result is a Fourier sine series with the coefficients b_n given by:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Example 1 – Determine the solution for the one-dimensional heat equation subjected to *u(0,t) = u(L,t) = 0*, if the initial conditions are given by *u(x,0) = f(x) = 4·(x/L)· (1-x/L)*. Use values of *k=1* and *L=1*.

First, we define a function *w(x,n,L)* that constitutes the integrand for the Fourier series coefficients:

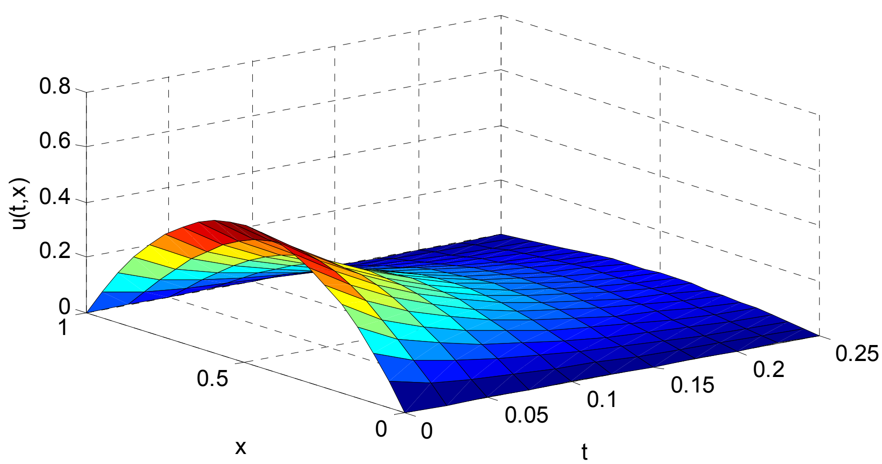
```
» w = inline('4*x/L.*(1-x/L).*sin(n*pi*x/L)','x','n','L')

w =

     Inline function:
     w(x,n,L) = 4*x/L.*(1-x/L).*sin(n*pi*x/L)
```

Next, we define the values of *L* and *k*, and load the vector of coefficients *b(n)*, which we call *bb*:

```
» L=1;bb = []; for n = 1:40, bb = [bb quad8(w,0,L,[],[],n,L)]; end;
```

The function *u(t,x)* is defined in the file *uheat.m* (see below). Following, we calculate values of the function in the ranges *0<x<1*, *0<t<0.25*, and produce a three dimensional plot of *u(t,x)*:

```
» xx = [0:0.05:1]; tt = [0:0.025:0.25]; nx = length(xx); nt = length(tt);

» for i = 1:nt
         for j = 1:nn
             uu(i,j) = uheat(k,L,bb,40,tt(i),xx(j));
         end;
  end;

» surf(tt,xx,uu');xlabel('t');ylabel('x');zlabel('u(t,x)');
```

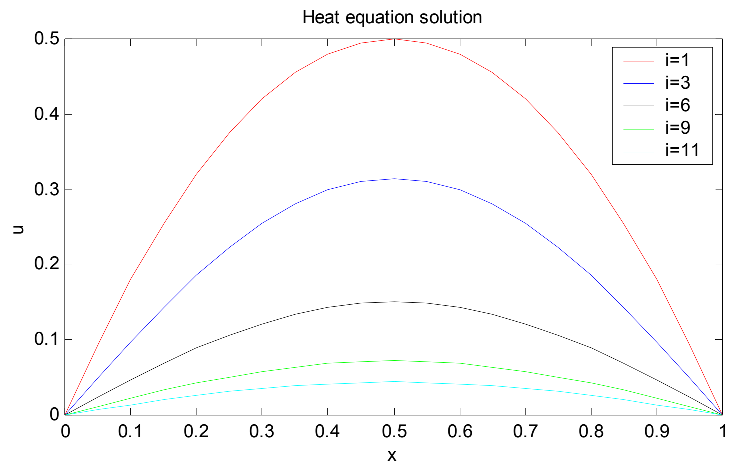


Function *uheat.m* is listed next:

```
function [uu] = uheat(k,L,b,n,t,x)
% Calculates solution for heat in a bar as functions of t,x
uu = 0;
for j = 1:n
   uu = uu + b(j)*sin(j*pi*x/L)*exp(-j^2*pi^2*k*t/L);
end;
```

Plots of the functions $u(x,t_0) = f_0(x)$, for specific values of t (i.e., $t = t_0$) are shown in the following figure:

```
» plot(xx,uu(1,:),'r',xx,uu(3,:),'b',xx,uu(6,:),'k',xx,uu(9,:),'g',xx,uu(11,:),'c');
» xlabel('x');ylabel('u');legend('i=1','i=3','i=6','i=9','i=11');
» title('Heat equation solution');
```



Analytical solutions to hyperbolic equations: One-dimensional solution of the wave equation

The wave equation, shown below, can be used to model the displacement of an elastic string or the longitudinal vibration of a beam:

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \left(\frac{\partial^2}{\partial x^2} u(x,t) \right)$$

For the case of an elastic string, $c^2 = \frac{T}{\mu}$, where T is the constant tension in the string and μ is the mass per unit length of the string. For the case of longitudinal vibration of a beam, $c^2 = \frac{g\,E}{\rho}$, where g is the acceleration of gravity, E is the modulus of elasticity, and ρ is the density of the beam.

Suppose we solve the wave equation for a vibrating string of length L using separation of variables with the boundary conditions *u(0,t)* = *u(L,0)*= *0*. Also, the initial shape of the string is given by *u(x,0)* = *f(x)*, and the initial speed of the string is given by $\frac{\partial}{\partial t} u(x,t) = g(x)$ at $t = 0$. We postulate a solution of the form u(x,t) = X(x)T(t), and replace this result in the original PDE:

$$X(x)\left(\frac{\partial^2}{\partial t^2}T(t)\right)=c^2\left(\frac{\partial^2}{\partial x^2}X(x)\right)T(t)$$

Dividing both sides of the equation by u(x,t), we get:

$$\frac{\frac{\partial^2}{\partial t^2}T(t)}{T(t)}=\frac{c^2\left(\frac{\partial^2}{\partial x^2}X(x)\right)}{X(x)}$$

This equation is only possible if the two sides of the equations are equal to a constant, say, $-\alpha^2$. With this we can write two ODEs, one for each side of the equation WaveEqn1, i.e.,

$$\left(\frac{\partial^2}{\partial t^2}T(t)\right)+\alpha^2\,T(t)=0$$

$$\left(\frac{\partial^2}{\partial x^2}X(x)\right)+\frac{\alpha^2\,X(x)}{c^2}=0$$

The solutions to these equations are:

$$T(t)=\_C1\cos(\alpha\,t)+\_C2\sin(\alpha\,t)$$

$$X(x)=\_C1\sin\left(\frac{\alpha\,x}{c}\right)+\_C2\cos\left(\frac{\alpha\,x}{c}\right)$$

Note: the constants _C1 and _C2 in the two solutions are not the same. The boundary conditions *u(0,t) = u(L,t) = 0* translate into *X(0) = X(L) = 0*. With these boundary conditions, we can form the following algebraic equations:

$$\_C2 := 0$$

$$0=\_C1\sin\left(\frac{\alpha\,L}{c}\right)$$

We have an eigenvalue equation given by $\sin\left(\frac{\alpha\,L}{c}\right)=0$. The solution to this equation is $\frac{\alpha\,L}{c}$ = 0, π, $2\,\pi$,…, or α = 0, $\frac{\pi\,c}{L}$, $\frac{2\,\pi\,c}{L}$,…. Using only the positive values, we can write $\alpha=\frac{n\,\pi\,c}{L}$, n = 0,1,2,…. Next, using the value of α and _C2 = 1, we obtain an expression for X(x) as follows:

$$\alpha := \frac{n\,\pi\,c}{L}$$

$$Sol2 := X(x)=\sin\left(\frac{n\,\pi\,x}{L}\right)$$

Function T(t) gets written as:

$$T(t) = \_C1 \cos\left(\frac{n\pi c t}{L}\right) + \_C2 \sin\left(\frac{n\pi c t}{L}\right)$$

The function *u(x,t)* is now:

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(\_C1 \cos\left(\frac{n\pi c t}{L}\right) + \_C2 \sin\left(\frac{n\pi c t}{L}\right)\right)$$

Application of the initial conditions provide the following equations:

$$f(x) = \_C1 \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \frac{\sin\left(\frac{n\pi x}{L}\right)\_C2\, n\pi c}{L}$$

Since the last two equations need to apply for n = 0,1,2, ..., we recognize in them the equations that define Fourier series if we use _C1 = a_n , and _C2 = b_n , i.e.,

$$f(x) = a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \frac{\sin\left(\frac{n\pi x}{L}\right) b_n\, n\pi c}{L}$$

The equations defining coefficients a_n and b_n are given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \text{ , and } b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \text{ , for } n = 1, 2, 3, \ldots$$

The final solution is, therefore,

$$u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

Example 1 - Consider the case of a vibrating string with the initial displacement given by f(x) = $\frac{x}{L}\left(1 - \frac{x}{L}\right)$, and the initial velocity given by g(x) = $\left(\frac{x}{L}\right)^2\left(1 - \frac{x}{L}\right)$. The boundary conditions are *u(0,t) = 0, u(L,t) = 0*. Determine the solution u(x,t) for this problem using components of the resulting Fourier series for n = 1, 2, ..., 20, if *c* = *1* and *L* = *1*.

The next definitions correspond to functions *w1(x)*, the integrand for *a*, and *w2(x)*, the integrand for b_n:

```
» w1 = inline('x/L*(1-x/L)*sin(n*pi*x/L)','x','n','L')
```

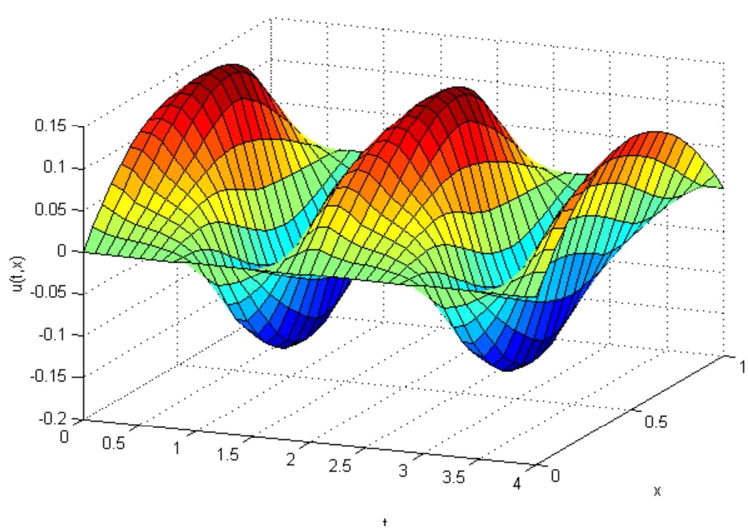
```
w1 =

     Inline function:
     w1(x,n,L) = x/L*(1-x/L)*sin(n*pi*x/L)

» w2 = inline('(x/L)^2*(1-x/L)*sin(n*pi*x/L)','x','n','L')

w2 =

     Inline function:
     w2(x,n,L) = (x/L)^2*(1-x/L)*sin(n*pi*x/L)
```

Next, we define the values of *L* and *c* and calculate the coefficients a_n and b_n. We also define the vectors of time and position values, namely, *tt* and *xx*, and calculate a matrix of values of *u(t,x)* with the function *uwave.m* (see below). We use the ranges $0<x<1$ and $0<t<4$:

```
» L = 1; c = 1;
» aa = []; for n = 1:20, aa = [aa quad8(w1,0,L,[],[],n,L)]; end;
» bb = []; for n = 1:20, bb = [bb quad8(w2,0,L,[],[],n,L)]; end;
» tt = [0:0.1:4]; xx = [0:0.05:1]; nt = length(tt); nx = length(xx);
» for i = 1:nt
          for j = 1:nx
               uu(i,j) = uwave(c,L,aa,bb,20,tt(i),xx(j));
          end;
     end;
```

A three dimensional plot of u(x,t) is shown next:

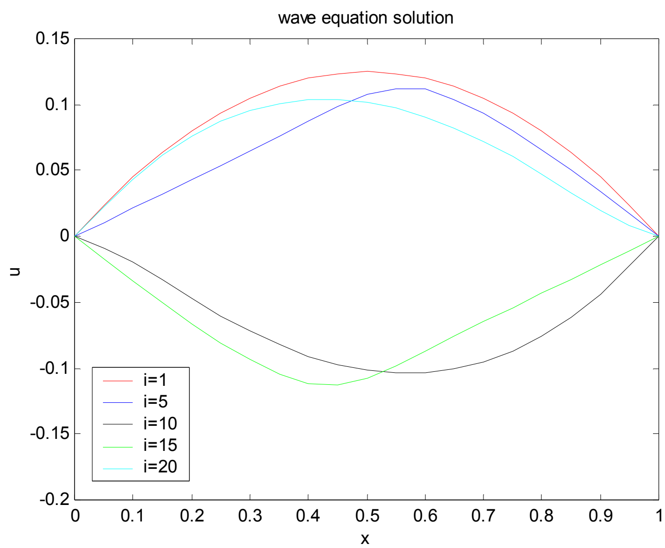
```
surf(tt,xx,uu');xlabel('t');ylabel('x');zlabel('u(t,x)');
```



An alternative way to present the result is to plot $u(x,t_0)$ vs. *x* for selected values of t_o as shown in the next SCILAB commands:

```
» plot(xx,uu(1,:),'r',xx,uu(5,:),'b',xx,uu(10,:),'k',xx,uu(15,:),'g',xx,uu(20,:),'c')
» xlabel('x');ylabel('u');title('wave equation solution');
» legend('i=1','i=5','i=10','i=15','i=20');
```

The plot thus generated is shown below:



This is a listing of function *uwave.m*:

```
function [uu] = uwave(c,L,a,b,n,t,x)
% Calculates solution for heat in a bar as functions of t,x
uu = 0;
for j = 1:n
   uu = uu + sin(j*pi*x/L)*(a(j)*cos(j*pi*c*t/L)+b(j)*sin(j*pi*c*t/L));
end;
```

Analytical solutions to elliptic equations: Two-dimensional solution to Laplace's equation in a rectangular domain.

Laplace's equation in two-dimensions is given by

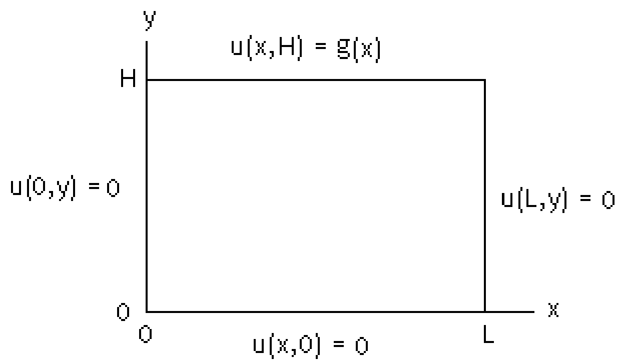
$$\left(\frac{\partial^2}{\partial x^2} u(x,y)\right) + \left(\frac{\partial^2}{\partial y^2} u(x,y)\right) = 0 .$$

In problems related to heat transfer, the two-dimensional Laplace equation describes the steady state distribution of temperature *u(x,y)* in the x-y plane. In fluid mechanics, *u(x,y)* could describe the velocity potential or the streamfunction for a two-dimensional potential flow. The problem requires two boundary conditions in the two independent variables x and y.

Laplace equation is solved in a rectangular domain so that 0<x<L, 0<y<H, a suitable set of boundary conditions may be

$$u(x,0) = 0 , \quad u(x,H) = g(x) , \quad u(0,y) = 0 , \quad u(L,y) = 0 .$$

as illustrated in the figure below:



Separation of variables suggests that we use a solution of the form,

$$u(x,y) = X(x)\,Y(y)\,.$$

Solving the equation through separation of variables proceeds in the following fashion:

$$\frac{\partial^2}{\partial x^2} u(x,y) = -\left(\frac{\partial^2}{\partial y^2} u(x,y)\right)$$

$$\left(\frac{\partial^2}{\partial x^2} X(x)\right) Y(y) = -X(x)\left(\frac{\partial^2}{\partial y^2} Y(y)\right)$$

Dividing by *u(x,y)* = *X(x)Y(y)* results in:

$$\frac{\frac{\partial^2}{\partial x^2} X(x)}{X(x)} = -\frac{\frac{\partial^2}{\partial y^2} Y(y)}{Y(y)}$$

For the two sides of the resulting equation to be equal, they both must be equal to a constant, λ^2, resulting in the following two ordinary differential equations:

$$\left(\frac{\partial^2}{\partial x^2} X(x)\right) + \lambda^2 X(x) = 0$$

$$\left(\frac{\partial^2}{\partial y^2} Y(y)\right) - \lambda^2 Y(y) = 0$$

The solution to the first ODE is:

$$X(x) = \_C1\,\sin(\lambda\,x) + \_C2\,\cos(\lambda\,x)$$

We next use the boundary conditions: $X(0) = 0$, and X(L) = 0, to determine the constants of integration:

_C2 = 0

$$0 = \_C1\,\sin(\lambda\,L)$$

This second equation produces an eigenvalue equation with the eigenvalues given by $\lambda = \frac{n\,\pi}{L}$.

$$\lambda := \frac{n\,\pi}{L}$$

The solution for X(x), with _C1 = 1 (since the value _C1 is arbitrary), is, therefore:

$$\mathrm{X}(x) = \cos\left(\frac{n\,\pi\,x}{L}\right)$$

The solution to the second ODE is:

$$\mathrm{Y}(y) = \_C3\,\cosh\left(\frac{n\,\pi\,y}{L}\right) + \_C4\,\sinh\left(\frac{n\,\pi\,y}{L}\right)$$

Utilizing the boundary condition: Y(0) = 0, we find for Y(y):

$$\_C3 := 0$$

$$\mathrm{Y}(y) = \_C4\,\sinh\left(\frac{n\,\pi\,y}{L}\right)$$

The product X(x)Y(y), which now depends on the value of the eigenvalue $\lambda_n = \frac{n\,\pi}{L}$, is referred to as $v_n(x, y) = \mathrm{X}(x)\,\mathrm{Y}(y)$, with the constant replaced by a_n:

$$v_n(x, y) = a_n\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\sinh\left(\frac{n\,\pi\,y}{L}\right)$$

The solution will be the sum of all possible functions $v_n(x, y)$, i.e.,

$$u := (x, y) \to \sum_{n=1}^{\infty} v_n(x, y)$$

The expression for the function $\mathrm{u}(x, y)$ is:

$$\mathrm{v}(x,y) = \sum_{n=1}^{\infty} a_n\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\sinh\left(\frac{n\,\pi\,y}{L}\right)$$

If we now evaluate the boundary condition, $\mathrm{u}(x, H) = \mathrm{g}(x)$, we find the following equation:

$$\sum_{n=1}^{\infty} a_n\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\sinh\left(\frac{n\,\pi\,H}{L}\right) = \mathrm{g}(x)$$

This result is a Fourier series expansion to $\mathrm{g}(x)$, such that the constants a_n are calculated by

$$a_n = 2\,\frac{\int_0^L \mathrm{g}(x)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,dx}{L\,\sinh\left(\frac{n\,\pi\,H}{L}\right)}$$

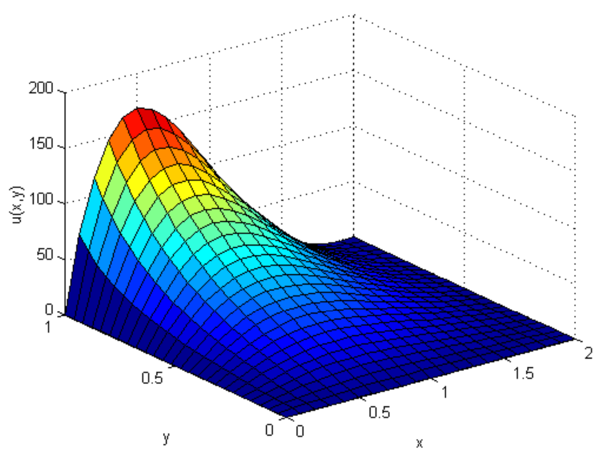
Example 1: Suppose that the dimensions of the solution domain are L = 2 and H = 1, and the boundary condition at y = H is given by $g(x) = 100\ x\ (L-x)^3$.

We start by defining function *w(x)*, the integrand for the Fourier series coefficients:

```
» w = inline('100*x.*(L-x).^3.*sin(n*pi*x/L)','x','n','L')

w =

     Inline function:
     w(x,n,L) = 100*x.*(L-x).^3.*sin(n*pi*x/L)
```

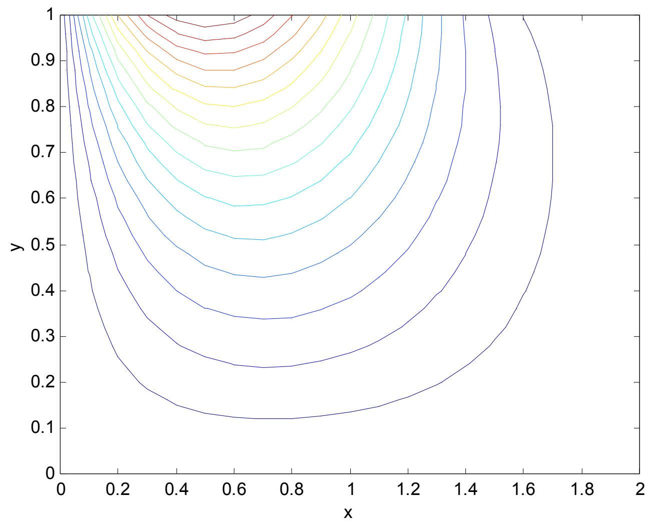
Next, we define the values of *L* and *H*, calculate 20 Fourier coefficients, and define and evaluate the solution *u(x,y)* in the ranges *0<x<L*, *0<y<H*. A three-dimensional plot of the function is shown. The function is contained in file *uLaplace1.m*.

```
» L = 2; H = 1;

» aa=[]; for j = 1:20, aa=[aa,2*quad8(w,0,L,[],[],j,L)/(L*sinh(j*pi*H/L))];end;

» xx=[0:L/20:L];yy=[0:H/20:H];nx=length(xx);ny=length(yy);
» uu1=zeros(nx,ny);
» for i = 1:nx
   for j = 1:ny
      uu1(i,j) = uLaplace1(L,H,aa,20,xx(i),yy(j));
   end;
end;


» surf(xx,yy,uu1);xlabel('x');ylabel('y');zlabel('u(x,y)');
```



Solutions to Laplace's equation in two-dimensions can also be represented by contour plots as shown below:

```
» contour(xx,yy,uu1',15);xlabel('x');ylabel('y');
```



Function *uLaplace1.m* is listed next:

```
function [uu] = uLaplace1(L,H,a,n,x,y)
% Laplace equation solution -- case 1
uu = 0;
for j = 1:n
   uu = uu + a(j)*sin(j*pi*x/L)*sinh(j*pi*y/L);
end;
```

More solutions to Laplace equation in a rectangular domain

The solution obtained above was facilitated by the use of zero boundary conditions in three of the boundaries. The zero boundary conditions at x = 0 and x = L produced the eigenvalues

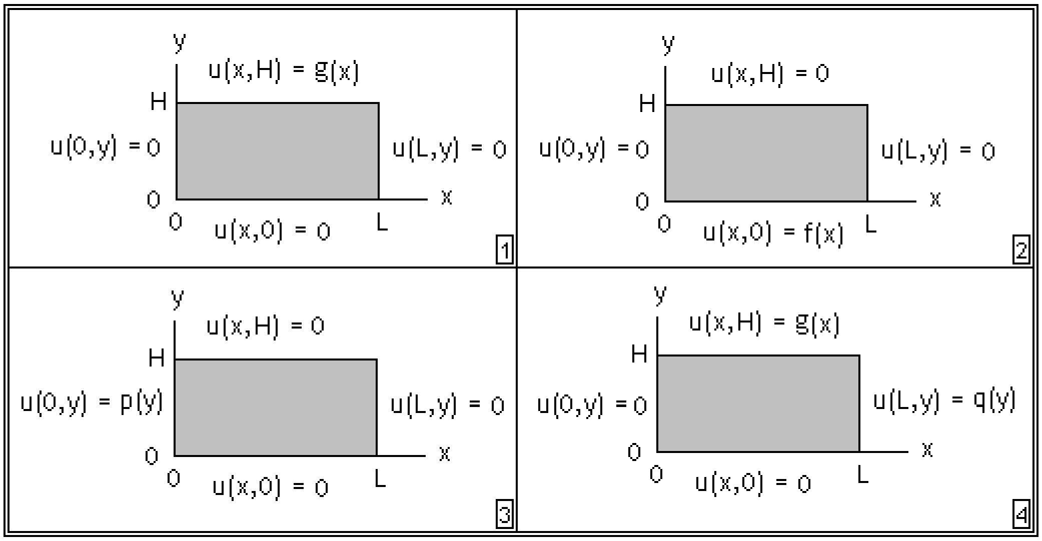
$$\lambda_n = \frac{n\,\pi}{L} \quad ,$$

while the zero boundary condition at y = 0 produced the series solution

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\,\pi\,x}{L}\right)\sinh\left(\frac{n\,\pi\,y}{L}\right).$$

Finally, the non-zero boundary condition at y = H, produced the coefficients a_n for the corresponding Fourier series. The case solved above is referred to as Case [1] in the figure below.

Solutions to the case of a rectangular domain where only one of the boundaries is non-zero can be found for cases [2] through [4] in the figure below using a procedure similar to that outlined above for case [1].



Because Laplace's equation is a linear equation, i.e., $\nabla^2(\phi_1 + \phi_2) = \nabla^2\phi_1 + \nabla^2\phi_2$, solutions with different boundary conditions, in the same domain, can be superimposed. Therefore, linear combinations of the solutions for the four cases illustrated in the figure above can be added to solve problems involving non-zero boundary conditions in more than one boundary. Suppose that $u_1(x,y)$, $u_2(x,y)$, $u_3(x,y)$, and $u_4(x,y)$, represent the solution for cases [1],[2],[3], and [4], respectively, in the figure above. The most general problem in the rectangular domain will use the following non-zero boundary conditions:

$$\mathrm{u}(x,0) = \alpha_1\,\mathrm{g}(x),\ \ \mathrm{u}(x,H) = \alpha_2\,\mathrm{f}(x),\ \ \mathrm{u}(0,y) = \alpha_3\,\mathrm{p}(y),\ \text{and}\ \ \mathrm{u}(L,y) = \alpha_4\,\mathrm{q}(y).$$

The solution will be obtained as

$$\mathrm{u}(x,y) = \alpha_1\,u_1(x,y) + \alpha_2\,u_2(x,y) + \alpha_3\,u_3(x,y) + \alpha_4\,u_4(x,y).$$

Solution to case [2]

To solve case [2] in the figure above, we start by using separation of variables, *u(x,y) = X(x)Y(y)*, on the Laplace equation:

$$\frac{\partial^2}{\partial x^2}\mathrm{u}(x,y) = -\left(\frac{\partial^2}{\partial y^2}\mathrm{u}(x,y)\right)$$

$$\left(\frac{\partial^2}{\partial x^2}\mathrm{X}(x)\right)\mathrm{Y}(y) = -\mathrm{X}(x)\left(\frac{\partial^2}{\partial y^2}\mathrm{Y}(y)\right)$$

Dividing by *u(x,y) = X(x)Y(y)* results in:

$$\frac{\frac{\partial^2}{\partial x^2}\mathrm{X}(x)}{\mathrm{X}(x)} = -\frac{\frac{\partial^2}{\partial y^2}\mathrm{Y}(y)}{\mathrm{Y}(y)}$$

For the two sides of the resulting equation to be equal, they both must be equal to a constant, λ^2, resulting in the following two ordinary differential equations:

$$\left(\frac{\partial^2}{\partial x^2} X(x)\right) + \lambda^2 X(x) = 0$$

$$\left(\frac{\partial^2}{\partial y^2} Y(y)\right) - \lambda^2 Y(y) = 0$$

The solution to the first ODE is:

$$X(x) = \_C1 \sin(\lambda\, x) + \_C2 \cos(\lambda\, x)$$

We next use the boundary conditions: $X(0) = 0$, and X(L) = 0, to determine the constants of integration:

$$0 = \_C2$$
$$0 = \_C1 \sin(\lambda x)$$

This second equation produces an eigenvalue equation with the eigenvalues given by $\lambda = \frac{n\,\pi}{L}$.
The solution for *X(x)*, with *_C2 = 1* (since the value _C2 is arbitrary), is, therefore:

$$X(x) = \cos\left(\frac{n\,\pi\,x}{L}\right)$$

The solution to the second ODE is:

$$Y(y) = \_C3 \cosh\left(\frac{n\,\pi\,y}{L}\right) + \_C4 \sinh\left(\frac{n\,\pi\,y}{L}\right)$$

Utilizing the boundary condition: Y(H) = 0, we find for Y(y):

$$0 = \_C3 \cosh\left(\frac{n\,\pi\,H}{L}\right) + \_C4 \sinh\left(\frac{n\,\pi\,H}{L}\right)$$

or

$$\_C4 := -\frac{\_C3 \cosh\left(\frac{n\,\pi\,H}{L}\right)}{\sinh\left(\frac{n\,\pi\,H}{L}\right)}$$

$$Y(y) = -\frac{\_C3\left(-\cosh\left(\frac{n\,\pi\,y}{L}\right)\sinh\left(\frac{n\,\pi\,H}{L}\right) + \cosh\left(\frac{n\,\pi\,H}{L}\right)\sinh\left(\frac{n\,\pi\,y}{L}\right)\right)}{\sinh\left(\frac{n\,\pi\,H}{L}\right)}$$

The product *X(x)Y(y),* which now depends on the value of the eigenvalue $\lambda_n = \frac{n\,\pi}{L}$, is referred to as $v_n(x, y) = X(x)\, Y(y)$, with the constant replaced by a_n:

$$v_n(x, y) = \frac{a_n \sin\left(\frac{n\,\pi\,x}{L}\right)\left(\cosh\left(\frac{n\,\pi\,y}{L}\right)\sinh\left(\frac{n\,\pi\,H}{L}\right) - \cosh\left(\frac{n\,\pi\,H}{L}\right)\sinh\left(\frac{n\,\pi\,y}{L}\right)\right)}{\sinh\left(\frac{n\,\pi\,H}{L}\right)}$$

The solution will be the sum of all possible functions $v_n(x,y)$, i.e.,

$$u(x,y) = \sum_{n=1}^{\infty} v_n(x,y)$$

The expression for the function $u(x,y)$ is:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{a_n \sin\left(\frac{n\pi x}{L}\right)\left(\cosh\left(\frac{n\pi y}{L}\right)\sinh\left(\frac{n\pi H}{L}\right) - \cosh\left(\frac{n\pi H}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)\right)}{\sinh\left(\frac{n\pi H}{L}\right)}$$

If we now evaluate the boundary condition, $u(x,0) = g(x)$, we find the following equation:

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

This result is a Fourier series expansion to $f(x)$, such that the constants a_n are calculated by

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Example 1: Suppose that the dimensions of the solution domain are L = 2 and H = 1, and the boundary condition at y = H is given by $f(x) = 100\ x\ (L-x)^3$.

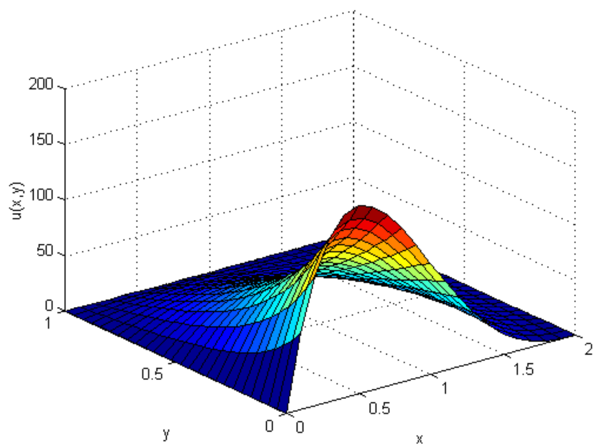
We start by defining function *w(x)*, the integrand for the Fourier series coefficients:

```
» w = inline('100*x.*(L-x).^3.*sin(n*pi*x/L)','x','n','L')

w =

     Inline function:
     w(x,n,L) = 100*x.*(L-x).^3.*sin(n*pi*x/L)
```

Next, we define the values of *L* and *H*, calculate 20 Fourier coefficients, and define and evaluate the solution *u(x,y)* in the ranges *0<x<L, 0<y<H*. A three-dimensional plot of the function is shown. The function is contained in file *uLaplace2.m*.

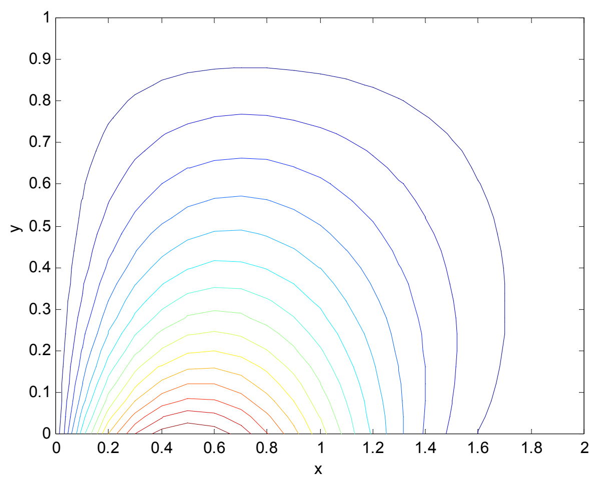
```
» L = 2; H = 1;

» aa = []; for j = 1:20, aa = [aa, 2*quad8(w,0,L,[],[],j,L)/L]; end;

» xx=[0:L/20:L];yy=[0:H/20:H];nx=length(xx);ny=length(yy);
» uu2=zeros(nx,ny);
» for i = 1:nx
   for j = 1:ny
      uu2(i,j) = uLaplace2(L,H,aa,20,xx(i),yy(j));
   end;
end;
```

```
» surf(xx,yy,uu2');xlabel('x');ylabel('y');zlabel('u(x,y)');
```



Solutions to Laplace's equation in two-dimensions can also be represented by contour plots as shown below:

```
» contour(xx,yy,uu2',15);xlabel('x');ylabel('y');
```

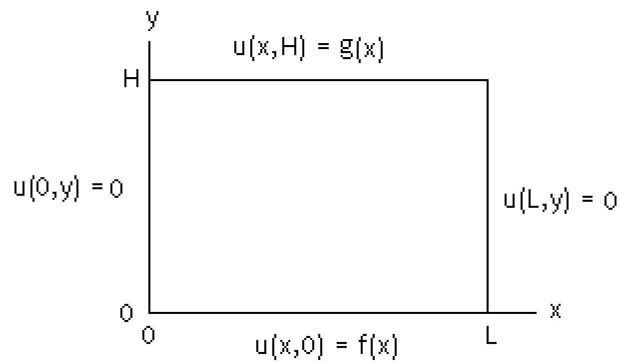


Function *uLaplace1.m* is listed next:

```
function [uu] = uLaplace1(L,H,a,n,x,y)
% Laplace equation solution -- case 1
uu = 0;
for j = 1:n
   uu = uu + a(j)*sin(j*pi*x/L)*sinh(j*pi*y/L);
end;
```

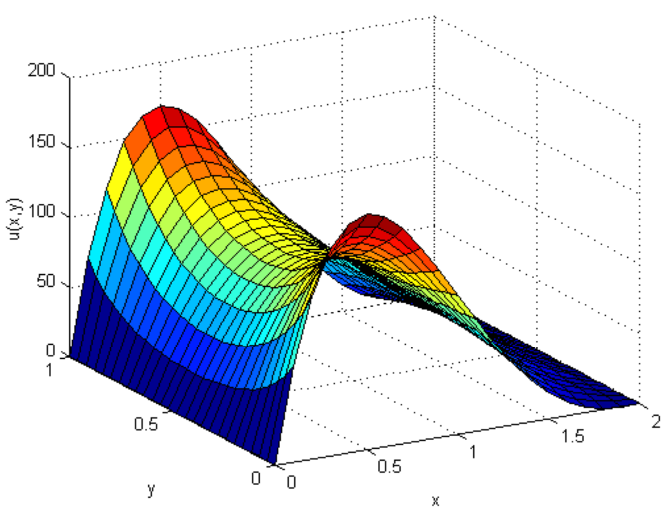
Superposition of solutions for cases [1] and [2]

The superposition of the solutions for cases [1] and [2] will be the solution to the problem whose boundary conditions are illustrated in the figure below:



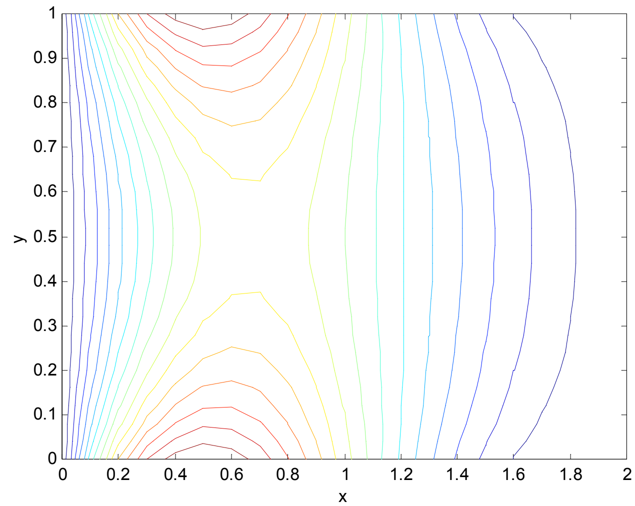
We use for the solution $f(x) = x\,(L-x)^3$, and $g(x) = x^3\,(L-x)$. Thus, we can simply add the results from cases 1 and 2, namely, *uu1+uu2*, and plot a surface plot of these results:

```
» uu = uu1 + uu2;
» surf(xx,yy,uu');xlabel('x');ylabel('y');zlabel('u(x,y)');
```



Using contours:

```
» contour(xx,yy,uu',15);xlabel('x');ylabel('y');
```

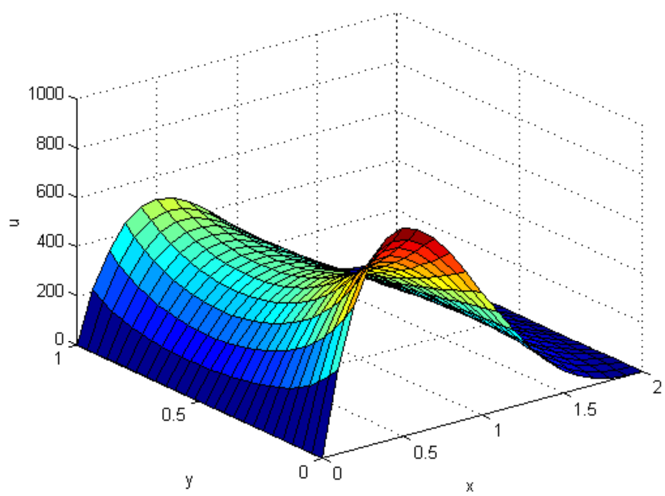


Suppose now, that the boundary conditions are such that $f_1(x) = 3\,\mathrm{f}(x)$ = $3\,x\,(L-x)^3$ and $g_1(x) = 5\,\mathrm{g}(x)$ = $5\,x^3\,(L-x)$. Since we already have the solutions corresponding to f(x) and g(x), we can obtain the solution to the problem with the new set of boundary conditions by constructing the following solution $\mathrm{u}(x,y) = 3\,u_1(x,y) + 5\,u_2(x,y)$:

```
>> uu=3*uu1+5*uu2;
```

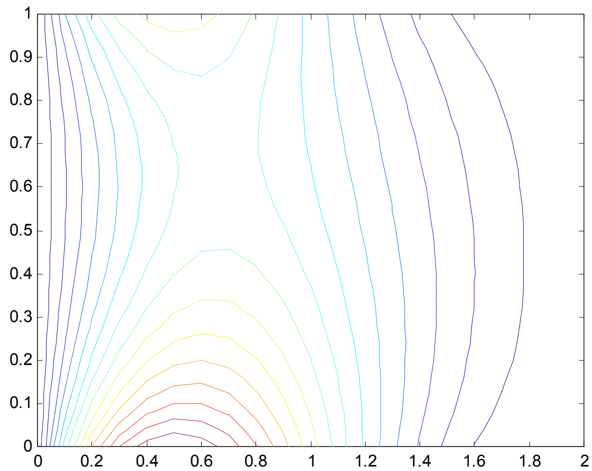
The three-dimensional plot corresponding to this solution is shown below:

```
-->plot3d(x,y,uu,45,45,'x@y@u(x,y)')
```



Contour plots of the solution are produced by using:

```
>> contour(x,y,uu,15)
```



Exercises

In the following exercises we use the following PDEs from physical science:

One-dimensional wave equation: $u_{tt} = c^2\, u_{xx}$

One-dimensional heat equation: $u_t = c^2\, u_{xx}$

Two-dimensional Laplace equation: $u_{xx}+u_{yy} = 0$

Two-dimensional Poisson equation: $u_{xx}+u_{yy} = f(x,y)$

With the notation $u_t = \partial u/\partial t$, $u_{xx} = \partial^2 u/\partial^2 x$, etc.

[1]. Determine if the following functions *u(x,t)* satisfy the one-dimensional wave equation:
(a) $u(x,t) = cxt$ (b) $u(x,t) = \ln(x^2-c^2t^2)$
(c) $u(x,t) = x^2 + c^2t^2$ (d) $u(x,t) = x/(x^2+c^2t^2)$

[2]. Determine if the following functions *u(x,t)* satisfy the one-dimensional heat equation:
(a) $u(x,t) = \exp(k(x+kc^2t))$ (b) $u(x,t) = t+x^2/(2c^2)$
(c) $u(x,t) = 2c^2/(2c^2t+x^2)$ (d) $u(x,t) = x^2 + 2c^2t^2$

[3]. Determine if the following functions *u(x,y)* satisfy the two-dimensional Laplace equation:
(a) $u(x,t) = k(x^2+y^2)$ (b) $u(x,t) = \tan^{-1}(y/x)$
(c) $u(x,t) = \ln(x^2+y^2)$ (d) $u(x,t) = x^3-3xy^2$

[4]. Determine if the functions *u(x,y)* satisfy the two-dimensional Poisson equation for the given function *f(x,y)*:

(a) $u(x,y) = x \cos y$, $f(x,y) = -x \cos y$
(b) $u(x,y) = x^2-2xy+y^2$, $f(x,y) = 4$
(c) $u(x,y) = x \ln y$, $f(x,y) = -x/y^2$
(d) $u(x,y) = (x+y)^3$, $f(x,y) = 12(x+y)$

[5]. Classify the following second-order partial differential equations as parabolic, hyperbolic, or elliptic:

(a) $u_t - u_{tx} - u_{xx} = 0$ (b) $u_{xx} - 2u_{xy} + u_{yy} = xy$
(c) $u_{xx} + u_{xt} - u_{tt} + u_x + u_t = 0$ (d) $u_{xx} + u_{yy} = 0$

Solve problems [5] through [13] using separation of variables and Fourier series expansions.

[6]. **Solution to the heat equation with derivative boundary conditions**. Using separation of variables, solve the one-dimensional heat equation for the temperature $u(x,t)$ on a bar of length $L = 1$, using as boundary conditions $\partial u/\partial x = 0$ for $x = 0$ and $x = L$. Assume that the initial temperature distribution is given by $u(x,0) = f(x) = x(1-x)$, and use $k=1$.

[7]. **Solution to the heat equation with heat convection loss .** Consider the one-dimensional heat equation including convection losses: $\partial u/\partial t = k\cdot(\partial^2 u/\partial x^2) - \beta u$, and subject to the boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$, and the initial condition: $u(x,0) = f(x)$.
(a) Using the transformation $u(x,t) = \exp(-\beta\cdot t)\cdot w(x,t)$, verify that the original PDE is transformed into $\partial w/\partial t = k\cdot(\partial^2 w/\partial x^2)$, and that the corresponding boundary and initial conditions transform into $w(0,t) = 0$, $w(L,t) = 0$, and $w(x,0) = f(x)$.
(b). For $L = 1$, $k = 1$, $\beta = 0.01$, and $f(x) = x(1-x)$ determine the solution $u(x,t)$ to the heat equation with convection loss.

[8]. **Solution to the diffusion-convection equation with constant flow velocity**. Many equations related to diffusion in moving fluids include a term involving the flow velocity v in the x-direction: $\partial u/\partial t + v\cdot(\partial u/\partial x) = k\cdot(\partial^2 u/\partial x^2)$. If v is a constant, this equation can be reduced to the form: $\partial w/\partial t = k\cdot(\partial^2 w/\partial x^2)$,by using the transformation

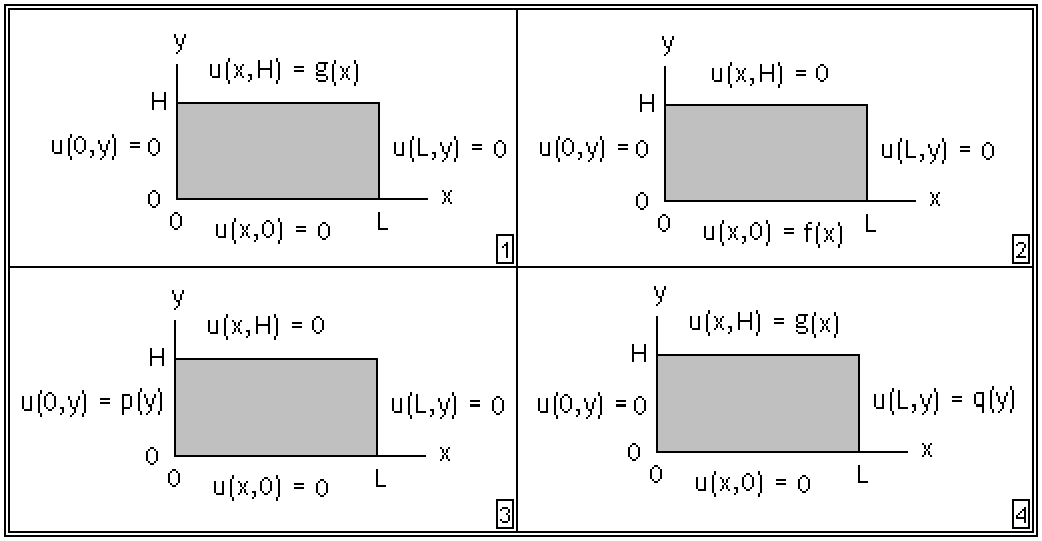
$$u(x,t) = \exp[\frac{v}{2\cdot k}\cdot(x - \frac{1}{2}\cdot v\cdot t)]\cdot w(x,t).$$

(a) Verify that the transformation indicated above produces the modified PDE: $\partial w/\partial t = k\cdot(\partial^2 w/\partial x^2)$.
(b) Solve the diffusion-convection equation with constant flow velocity in the domain $0<x<L$ for $L = 1$, $k = 1$, $v = 1$, and $k = 1$, subject to the boundary conditions $u(0,t) = 0$, and $u(L,t) = 0$, and initial conditions $u(x,0) = x(1-x)$.

[9]. Solve the one-dimensional wave equation for $c = 1$ in the domain $0 < x < L$, with $L = 1$, given the boundary conditions $u(0,t) = 0$, and $\partial u/\partial x = 0$ at x = L, and the initial condition $u(x,0) = f(x) = \cos(x)$.

[10]. Prove that u(x,t) = F(x+ct) + F(x-ct) is a solution to the one-dimensional wave equation. Plot the result if $F(x) = \exp(-x^2/2)$ with c = 1 in the interval -10 < x < 10, and -10 < t < 10. Produce a three-dimensional surface plot as well as an animation of the results.

[11]. Referring to the following figure, solve case [3] of Laplace's equation in the rectangular domain L = 2, H = 1, if p(y) = (y/H)(1-y/H). Produce a surface plot as well as a contour plot of the results.



[12]. Referring to the figure of problem [12], solve case [4] of Laplace's equation in the rectangular domain L = 2, H = 1, if q(y) = $(y/H)^2(1-y/H)$. Produce a surface plot as well as a contour plot of the results.

[13]. Using the results of problems [12] and [14] determine the solution to Laplace's equation in the rectangular domain L = 2, H = 1, if the boundary conditions are u(x,0) = u(x,H) = 0, u(0,y) = p(y) = (y/H)(1-y/H), and u(L,y) = q(y) = $(y/H)^2(1-y/H)$. Produce a surface plot as well as a contour plot of the results.