Name. KEY

Partial Differential Equations 3150

Sample Final Exam

Final Exam Date: Thursday, 2 May 2013

Instructions: This exam is timed for 120 minutes. You will be given 30 extra minutes to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

Problems 1,2,3: Selected problem types from the first two midterm exams. See the sample exams and exam solutions for midterms 1 and 2, for specific examples.

4. (CH4. Poisson Problem)
Solve for $u(x,y)$ in the Poisson problem

$$\begin{cases} u_{xx}+u_{yy} = \sin(\pi x)\sin(\pi y), & 0<x<1,\ 0<y<1, \\ u(x,0) = 4\sin(\pi x), & 0<x<1, \\ u(x,y) = 0 & \text{on the other 3 boundary edges.} \end{cases}$$

This problem is studied in Asmar section 4.6 for the disk and in Section 3.9 for a rectangular plate. Both use the method of eigenfunctions and the Helmholtz equation $\nabla^2\phi = -\lambda\phi$

Sub-divide. Let $u = u_1 + u_2$, where

u_1 solves $\begin{cases} u_{xx}+u_{yy} = f(x,y) \\ u(x,y)=0 \text{ on boundary} \end{cases}$; u_2 solves $\begin{cases} u_{xx}+u_{yy}=0 \\ u(x,0) = 4\sin(\pi x) \\ u=0 \text{ on 3 edges} \end{cases}$

Find u_2

$$u_2 = \sum_1^\infty A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}(b-y)\right) \quad \text{where } a=b=1$$

$$A_n = \frac{2}{\sinh(n\pi)}\int_0^1 4\sin(\pi x)\sin(n\pi x)\,dx$$

$A_n = 0$ except for $n=1$, $A_1 = \dfrac{4}{\sinh(\pi)}$, $\boxed{u_2 = \dfrac{4\sin(\pi x)}{\sinh(\pi)}\sinh(\pi(1-y))}$

Find u_1

$$u_1 = \sum_1^\infty\sum_1^\infty E_{mn} \underbrace{\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)}_{\text{product solutions}} \quad \text{where } a=b=1$$

$$f(x,y) = \sum\sum -E_{mn}\lambda_{mn}\sin(n\pi x)\sin(m\pi y), \quad \lambda_{mn} = (m\pi)^2+(n\pi)^2$$

By orthogonality and $f(x,y) = \sin(\pi x)\sin(\pi y)$, only $m=n=1$ produce a nonzero answer $-E_{mn}\lambda_{mn}$. Then $-E_{11}\lambda_{11} = 1$ and

$$\boxed{u_1 = \frac{-1}{2\pi^2}\sin(\pi x)\sin(\pi y)}$$

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Then $\boxed{u = u_1 + u_2}$

Answer Check: u_2 checks because $(\sin(\pi x))'' = -\pi^2\sin\pi x$ and $(\sinh(\pi(1-y)))'' = \pi^2\sinh(\pi(1-y))$. To check u_1, take 2nd derivatives and add to $= (-\pi^2 u_1 - \pi^2 u_1) = f(x,y)$. The BC of each are checked also.

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5. (CH7. Fourier Transform: Infinite Rod)
Let $f(x) = \textbf{pulse}(x, -2, 2)$, that is, $f(x) = 1$ for $|x| < 2$ and $f(x) = 0$ elsewhere on $-\infty < x < \infty$.

(a) [75%] Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases}$$

See Asmar Ex 1, Section 7.4.

(b) [25%] Evaluate the limit of $u(x,t)$ at $x = \infty$ and $x = -\infty$.

(a) Let $y(t) = FT[u(x,t)]$. Then Fourier transform rules give

$$FT[u_t] = FT[u_{xx}]$$
$$\frac{d}{dt} FT[u] = (i\omega)^2 FT[u]$$
$$\begin{cases} \frac{d}{dt} y(t) = -\omega^2 y(t) \\ y(0) = F(\omega) = \text{Fourier transform of } f(x) \end{cases}$$

By the integrating factor shortcut, $y(t) = \frac{c}{e^{\omega^2 t}}$ and $C = F(\omega)$, which implies $y(t) = F(\omega)e^{-\omega^2 t}$. This method continues, writing $e^{-\omega^2 t} = FT[g]$ and then $y(t) = F(\omega)\,G(\omega) = FT[f * g]$. (Details left out)

(a) Heat Kernel method

$$g_t = \frac{1}{\sqrt{2t}} e^{-x^2/(4t)} \quad \text{and} \quad u(x,t) = (f * g_t)(x)$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, g_t(x-s)\,ds = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} 1 \cdot g_t(x-s)\,ds$$
$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-2}^{2} e^{-\frac{(x-s)^2}{4t}} ds \right) \cdot \frac{1}{\sqrt{2t}}$$

change vars

$$z^2 = \frac{(x-s)^2}{4t} \quad \text{or} \quad z = (x-s)/(2\sqrt{t}), \quad dz = -ds/(2\sqrt{t})$$

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \left(\int_{z_1}^{z_2} e^{-z^2} dz \right) (2\sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\pi}} \left(\operatorname{erf}(z_2) - \operatorname{erf}(z_1) \right)$$

where $z_1 = \frac{x-2}{2\sqrt{t}}$, $z_2 = \frac{x+2}{2\sqrt{t}}$

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(b) limit Because $\operatorname{erf}(\infty) = 1$ and $\operatorname{erf}(-\infty) = -1$ then $u(\infty, t) = 0 = u(-\infty, t)$.

6. (CH4. Heat Equation and Gauss' Heat Kernel)
Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= \frac{1}{4}u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -1 < x < 1, \\ f(x) &= \begin{cases} 20 & |x| < 1, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Hint: Use convolutions, the heat kernel, the error function $\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$, and Fourier transform rules to solve the problem. The answer is expressed in terms of the error function.

The heat kernel is $g_t = \sqrt{\frac{2}{t}}\, e^{-x^2/t}$ because $c^2 = \frac{1}{4}$, $c = \frac{1}{2}$

Then $u(x,t) = (f * g_t)(x)$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)\, g_t(x-s)\,ds$$

$$= \frac{20}{\sqrt{2\pi}}\int_{-1}^{1} g_t(x-s)\,ds \qquad \text{use def of } f(x)$$

$$= \frac{20}{\sqrt{2\pi}}\sqrt{\frac{2}{t}}\int_{-1}^{1} e^{-(x-s)^2/t}\,ds$$

$$= \frac{20}{\sqrt{\pi t}}\left(\int_{z_1}^{z_2} e^{-z^2}dz\right)\sqrt{t} \qquad \text{Let } z = \frac{x-s}{\sqrt{t}}, \quad dz = -\frac{ds}{\sqrt{t}}$$

$$= \frac{20}{\sqrt{\pi}}\left(\mathrm{erf}(z_2) - \mathrm{erf}(z_1)\right) \qquad z_1 = \frac{x-1}{\sqrt{t}}, \quad z_2 = \frac{x+1}{\sqrt{t}}$$

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