

# Partial Differential Equations 3150

S2013

## Midterm Exam 1 Problem List

- Problem 1. (**Vibration of a Finite String**)
- Problem 2. (**Periodic Functions**)
- Problem 3. (**Fourier Series and Gibb's Overshoot**)
- Problem 4. (**Cosine and Sine Series**)
- Problem 5. (**Convergence of Fourier Series**)

See the posted solutions for Exam 1, at the course web site.

## Midterm Exam 2 Problem List

- Problem 1. The exam will have one problem, either 1a or 1b.
  - \* Problem 1a. (**CH3. Finite String: Fourier Series Solution**)
  - \* Problem 1b. (**CH3. d'Alembert's Solution: Finite String**)
- Problem 2. (**CH3. Heat Conduction in a Bar**)
- Problem 3. (**CH4. Rectangular Membrane**)
- Problem 4. (**CH4. Steady-State Heat Conduction on a Disk**)

Solutions to Exam 2 will be posted before the final exam.

## Final Exam Problem List

- Problem 1. Selected from Exams 1 and 2.
- Problem 2. Selected from Exams 1 and 2.
- Problem 3. Selected from Exams 1 and 2.
- Problem 4. (**CH4. Poisson Problem**)
- Problem 5. (**CH7. Fourier Transform: Infinite Rod**)
- Problem 6. (**CH7. Heat Equation and Gauss' Heat Kernel**)

Solutions to the 2009 final are posted at the course web site. Problem 6 is new, it was not on the 2009 exam. The sample is Exercise 1, section 7.4 of Asmar.

**Partial Differential Equations 3150**

Sample Midterm Exam 2

Exam Date: Monday, 22 April 2013

**Instructions:** This exam is timed for 50 minutes. You will be given double time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 to 4 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

**1a. (CH3. Finite String: Fourier Series Solution)**

(a) [75%] Display the series formula without derivation details for the finite string problem

$$\begin{cases} u_{tt}(x, t) = c^2 u_{xx}(x, t), & 0 < x < L, & t > 0, \\ u(0, t) = 0, & & t > 0, \\ u(L, t) = 0, & & t > 0, \\ u(x, 0) = 0, & 0 < x < L, \\ u_t(x, 0) = g(x), & 0 < x < L. \end{cases}$$

(b) [25%] Display an explicit formula for the Fourier coefficients which contains the symbols  $L$ ,  $g(x)$ .

Use this page to start your solution. Attach extra pages as needed, then staple.

**1b. (CH3. d'Alembert's Solution: Finite String)**

Let  $f(x) = \begin{cases} 0.3x & 0 \leq x \leq 0.5, \\ 0.3(1-x) & 0.5 < x \leq 1. \end{cases}$  and define  $g(x) = 0$  on  $0 \leq x \leq 1$ . Assumed is d'Alembert's solution to the vibrating string problem on  $0 < x < 1$ ,  $t > 0$ , which is the formula

$$u(x, t) = \frac{1}{2} (f^*(x - ct) + f^*(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Assume  $L = 1$  and  $c = 1/\pi$ .

- (a) [25%] Define  $f^*$ ,  $g^*$  as appropriate periodic extensions of  $f$  and  $g$ , respectively.
- (b) [75%] Display a formula for  $u(x, \pi/3)$  on  $0 < x < 1$ , defined in terms of  $f^*$  and  $g^*$ .

Use this page to start your solution. Attach extra pages as needed, then staple.

**2. (CH3. Heat Conduction in a Bar)**

Consider the heat conduction problem in a laterally insulated bar of length 1 with one end at zero Celsius and the other end at 100 Celsius. The initial temperature along the bar is given by function  $f(x)$ .

$$\begin{cases} u_t &= c^2 u_{xx}, & 0 < x < 1, & t > 0, \\ u(0, t) &= 0, & & t > 0, \\ u(1, t) &= 100, & & t > 0, \\ u(x, 0) &= f(x), & 0 < x < 1. & \end{cases}$$

(a) [25%] Find the steady-state temperature  $u_1(x)$ .

(b) [50%] Solve the bar problem with zero Celsius temperatures at both ends, but  $f(x)$  replaced by  $f(x) - u_1(x)$ . Call the answer  $u_2(x, t)$ . The answer has Fourier coefficients in integral form, unevaluated, to save time.

(c) [25%] Explain why  $u(x, t) = u_1(x) + u_2(x, t)$ .

Use this page to start your solution. Attach extra pages as needed, then staple.

**3. (CH4. Rectangular Membrane)**

Consider the general membrane problem

$$\begin{cases} u_{tt}(x, y, t) = c^2 (u_{xx}(x, y, t) + u_{yy}(x, y, t)), & 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ u(x, y, t) = 0 & \text{on the boundary,} \\ u(x, y, 0) = f(x, y), & 0 < x < a, \quad 0 < y < b, \\ u_t(x, y, 0) = g(x, y), & 0 < x < a, \quad 0 < y < b. \end{cases}$$

Solve the problem for  $a = b = 1$ ,  $c = 1/\pi$ ,  $f(x, y) = 0$ ,  $g(x, y) = 1$ .Alternate problem type: Replace  $f = 0, g = 1$  by  $f = 1, g = 0$ .

Use this page to start your solution. Attach extra pages as needed, then staple.

**4. (CH4. Steady-State Heat Conduction on a Disk)**

Consider the problem

$$\begin{cases} u_{rr}(r, \theta) + \frac{1}{r}u_r(r, \theta) + \frac{1}{r^2}u_{\theta\theta}(r, \theta) = 0, & 0 < r < a, \quad 0 < \theta < 2\pi, \\ u(a, \theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

Solve for  $u(r, \theta)$  when  $a = 1$  and  $f(\theta) = 100 \mathbf{pulse}(\theta, 0, \pi)$ , that is,  $f(\theta) = 100$  on  $0 \leq \theta < \pi$ ,  $f(\theta) = 0$  on  $\pi \leq \theta < 2\pi$ .

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**Partial Differential Equations 3150**

Sample Final Exam

Final Exam Date: Thursday, 2 May 2013

**Instructions:** This exam is timed for 120 minutes. You will be given 30 extra minutes to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

**Problems 1,2,3:** Selected problem types from the first two midterm exams. See the sample exams and exam solutions for midterms 1 and 2, for specific examples.

**4. (CH4. Poisson Problem)**Solve for  $u(x, y)$  in the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} = \sin(\pi x) \sin(\pi y), & 0 < x < 1, \quad 0 < y < 1, \\ u(x, 0) = 4 \sin(\pi x), & 0 < x < 1, \\ u(x, y) = 0 & \text{on the other 3 boundary edges.} \end{cases}$$

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**5. (CH7. Fourier Transform: Infinite Rod)**

Let  $f(x) = \text{pulse}(x, -2, 2)$ , that is,  $f(x) = 1$  for  $|x| < 2$  and  $f(x) = 0$  elsewhere on  $-\infty < x < \infty$ .

(a) [75%] Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x, t) = u_{xx}(x, t), & -\infty < x < \infty, \quad t > 0, \\ u(x, 0) = f(x), & -\infty < x < \infty. \end{cases}$$

(b) [25%] Evaluate the limit of  $u(x, t)$  at  $x = \infty$  and  $x = -\infty$ .

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**6. (CH4. Heat Equation and Gauss' Heat Kernel)**

Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x, t) = \frac{1}{4}u_{xx}(x, t), & -\infty < x < \infty, \quad t > 0, \\ u(x, 0) = f(x), & -1 < x < 1, \\ f(x) = \begin{cases} 20 & |x| < 1, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

**Hint:** Use convolutions, the heat kernel, the error function  $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ , and Fourier transform rules to solve the problem. The answer is expressed in terms of the error function.

Use this page to start your solution. Attach extra pages as needed, then staple.