KEY

Partial Differential Equations 3150

Midterm Exam 2

Exam Date: Monday, 22 April 2013

Instructions: This exam is timed for 50 minutes. You will be given extra time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 to 4 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(CH3. Finite String: Fourier Series Solution)**

(a) [75%] Display the series formula without derivation details for the finite string problem

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & 0 < x < L, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L, \\ u_t(x,0) = g(x), & 0 < x < L. \end{cases}$$

Symbols f and g should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

(b) [25%] Display an explicit formula for the Fourier coefficients which contains the symbols L, $f(x)$, $g(x)$.

(a) Normal modes: $\sin(n\pi x/L)\cos(n\pi c t/L)$, $\sin(n\pi x/L)\sin(n\pi c t/L)$

ans 1 $\longrightarrow$ $u(x,t)$ = superposition of the normal modes

$$= \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)\cos(n\pi c t/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\sin(n\pi c t/L)$$

(b) $f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$

ans 2 $\longrightarrow$ $a_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$ by $\perp$ relations for $\{\sin(n\pi x/L)\}_{n=1}^{\infty}$

$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin(n\pi x/L)$

ans 3 $\longrightarrow$ $b_n = \frac{L}{n\pi c}\frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)\,dx$ by $\perp$ relations

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2. (CH3. Heat Conduction in a Bar) 96

Consider the heat conduction problem in a laterally insulated bar of length 2 with one end at 50 Celsius and the other end at zero Celsius. The initial temperature along the bar is given by function $f(x)$, which is a symbol used throughout the problem, devoid of a specific formula.

$$\begin{cases} u_t &= c^2 u_{xx}, & 0<x<2, & t>0, \\ u(0,t) &= 50, & & t>0, \\ u(2,t) &= 0, & & t>0, \\ u(x,0) &= f(x), & 0<x<2. & \end{cases}$$

A (a) [25%] Find the steady-state temperature $u_1(x)$.

A (b) [50%] Solve the bar problem with zero Celsius temperatures at both ends, but $f(x)$ replaced by $f(x) - u_1(x)$. Call the answer $u_2(x,t)$. The answer has Fourier coefficients in integral form, unevaluated to save time.

B −4 (c) [25%] Display an answer check for the solution $u(x,t) = u_1(x) + u_2(x,t)$.

a)

$$u_1 = \left[\frac{u(2,t) - u(0,t)}{L}\right]x + u(0,t) = \left[\frac{0-50}{2}\right]x + 50$$

$$= -25x + 50 = \boxed{25(2-x)}$$

b)

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = 50 \\ u(1,t) = 0 \\ u(x,0) = f(x) - u_1(x) = u_2(x,t) \end{cases}$$

$$u_2(x,t) = X(x)\,T(t) \qquad T = Ke^{-\lambda_n t} \qquad \lambda_n = \left(\frac{c\pi n}{2}\right)^2$$

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\lambda_n t}$$

with $b_n = 2\int_0^2 [f(x) - u_1] \sin\left(\frac{n\pi x}{2}\right) dx$

turn →

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c) $u(x,t) = u_1(x) + u_2(x,t)$

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = 50 \\ u(2,t) = 0 \\ u(x,0) = f(x) \end{cases} = \begin{cases} \overset{u_{xx}=0}{\cancel{u_t = c^2 u_{xx}}} \\ u(0,t) = 50 \\ u(2,t) = 0 \\ u(x,0) = 25(2-x) \end{cases} + \begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = 0 \\ u(2,t) = 0 \\ u(x,0) = f(x) - 25(2-x) \end{cases}$$

Thus, it can be seen from superposition that $u(x,t)$ does indeed equal $u_1(x) + u_2(x,t)$

Details for DE expected

Details Given $u = u_1 + u_2$. Verify u is a solution to the BVP.

$$\partial_t u = \partial_t u_1 + \partial_t u_2 = 0 + c^2 \partial_x \partial_x u_2 \qquad \Bigg| \qquad \partial_x \partial_x u = \partial_x^2 u_1 + \partial_x^2 u_2 = 0 + \partial_x^2 u_2$$

$\Rightarrow \partial_t u = c^2 \partial_x \partial_x u$, PDE verified

$u(0,t) = u_1(0) + u_2(0,t) = 50 + 0 = 50$ First BC verified

$u(2,t) = u_1(2) + u_2(2,t) = 0 + 0 = 0$ second BC verified

$u(x,0) = u_1(x) + u_2(x,0) = u_1(x) + f(x) - u_1(x)$
$= f(x)$ Third condition verified.

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3. (CH4. Rectangular Membrane)
Consider the general membrane problem

$$\begin{cases} u_{tt}(x,y,t) &= c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a,\quad 0<y<b,\quad t>0,\\ u(x,y,t) &= 0 & \text{on the boundary,}\\ u(x,y,0) &= f(x,y), & 0<x<a,\quad 0<y<b,\\ u_t(x,y,0) &= g(x,y), & 0<x<a,\quad 0<y<b. \end{cases}$$

Solve the problem for $a=b=c=1$, $f(x,y)=1$, $g(x,y)=0$. Expected are displays for the **normal modes**, a **superposition** formula for $u(x,y,t)$, and **explicit numerical values** for the generalized Fourier coefficients.

The solution is a superposition of the normal modes obtained from separation of variables as:

$$\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right]$$

with $\lambda_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right]$$

Where the Fourier coefficients are:

$$B_{mn}=4\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$\lambda_{mn}B^*_{mn}=4\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

turn →
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'roblem 3 continued

plugging in:

$a=b=c=1 \quad f(x,y)=1 \quad g(x,y)=0$

normal modes:

$$\sin(m\pi x)\sin(n\pi y)\left[B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right]$$

$$\lambda_{mn}=\pi\sqrt{m^2+n^2}$$

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(m\pi x)\sin(n\pi y)\left[B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right]$$

Where the fourier coefficient are:

$$B_{mn}=4\int_0^1\int_0^1 \sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$=4\int_0^1 -\cos(m\pi x)\sin(n\pi y)\left(\frac{1}{m\pi}\right)\Big|_0^1 dy$$

$$=4\int_0^1\left[-\cos(m\pi)\sin(n\pi y)\left(\frac{1}{m\pi}\right)-\sin(n\pi y)\left(\frac{1}{m\pi}\right)\right]dy$$

$$=4\left[\cos(m\pi)\cos(n\pi y)\left(\frac{1}{mn\pi^2}\right)+\cos(n\pi y)\left(\frac{1}{mn\pi^2}\right)\right]\Big|_0^1$$

$$B_{mn}=4\left[\cos(m\pi)\cos(n\pi)\left(\frac{1}{mn\pi^2}\right)+\cos(n\pi)\left(\frac{1}{mn\pi^2}\right)-\cos(m\pi)\left(\frac{1}{mn\pi^2}\right)-\left(\frac{1}{mn\pi^2}\right)\right]$$

$$B_{mn}=\frac{4}{mn\pi^2}\left[\cos(m\pi)\cos(n\pi)+\cos(n\pi)-\cos(m\pi)-1\right]$$

turn →

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Problem 3 continued

$$\lambda_{mn} B_{mn}^{*} = 4\int_0^1\int_0^1 0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) dx\, dy$$

$$= 4\int_0^1\int_0^1 0 \, dx\, dy = 0$$

$$\boxed{B_{mn}^{*} = 0}$$

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4. (CH4. Steady-State Heat Conduction on a Disk)
Consider the heat conduction problem on a disk

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$$\begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 0 < r < a, \quad 0 < \theta < 2\pi, \\ u(a,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

Solve for $u(r,\theta)$ when $a = 1$ and $f(\theta) = \textbf{pulse}(\theta, 0, \pi/2)$, that is, $f(\theta) = 1$ on $0 \le \theta < \pi/2$, $f(\theta) = 0$ on $\pi/2 \le \theta < 2\pi$.

$R(r)\Theta(\theta) \rightarrow$ product solution

$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$ dividing by product solution and mutiplying by r^2

$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$ where $\frac{\Theta''}{\Theta} = -\lambda$

$r^2R'' + rR' - \lambda R = 0$

$$\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}$$

$R(r) = c_1 r^n + c_2 r^{-n}$

Because R is bounded $\rightarrow c_2 = 0 \Rightarrow R(r) = c_1 r^n$

Product solutions
$$\begin{cases} u = \left(\frac{r}{a}\right)^n \cos(n\theta) & n \ge 0 \Rightarrow r^n \cos n\theta \\ u = \left(\frac{r}{a}\right)^n \sin(n\theta) & n > 0 \Rightarrow r^n \sin n\theta \end{cases}$$

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Problem 4 continued

from the principal of superposition: 1

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] \left(\frac{r}{a}\right)^n$$

with orthogonal set S: 1, sin(nθ), cos(nθ)

knowing the rules of orthogonality:

$$\begin{cases} \int_a^b f \cdot g = 0 & \text{for } f \text{ and } g \text{ is } S \\ \int_a^b f^2 > 0 & \text{for } f \text{ in } S \end{cases}$$

Thus, it is obvious that

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{\pi/2} 1\, d\theta = \frac{1}{2\pi}\theta \Big|_0^{\pi/2} = \frac{\pi}{4\pi} = \boxed{\frac{1}{4}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi/2} 1\cos(n\theta)\, d\theta = \frac{1}{n\pi}\sin(n\theta)\Big|_0^{\pi/2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi/2} 1\sin(n\theta)\, d\theta = \frac{1}{n\pi}(-\cos(n\theta))\Big|_0^{\pi/2}$$

$$\rightarrow a_n = \frac{1}{n\pi}\sin\left(\frac{\pi}{2}n\right)$$

$$\rightarrow b_n = \frac{1}{n\pi}\left(-\cos\left(\frac{\pi}{2}n\right) + \cancel{\cos}^{\,1}(0)\right) = \frac{1}{n\pi}\left[\cos\left(\frac{\pi}{2}n\right) + 1\right]$$

$$u(r,\theta) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\sin\left(\frac{\pi}{2}n\right)\cos n\theta + \frac{1}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + 1\right]\sin(n\theta)\right](r)^{n}$$