Name ________________________________

Partial Differential Equations 3150

Midterm Exam 1

Exam Date: Wednesday, 27 February

Instructions: This exam is timed for 50 minutes. Up to 60 minutes is possible. No calculators, notes, tables or books. Problems use only chapters 1 and 2 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Vibration of a Finite String)

The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(2,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= -11\sin(5\pi x). \end{aligned}$$

Answer:

Because the wave initial shape is zero, then the only normal modes are sine times sine. The initial wave velocity is already a Fourier series, using orthogonal set $\{\sin(n\pi x/2)\}_{n=1}^{\infty}$. The 1-term Fourier series $-11\sin(5\pi x)$ can be modified into a solution by inserting the missing sine factor present in the corresponding normal mode. Then $u(x,t) = -11\sin(5\pi x)\sin(5\pi ct)/(5\pi)$. We check it is a solution.

2. (Periodic Functions)

(a) [30%] Find the period of $f(x) = \sin(x)\cos(2x) + \sin(2x)\cos(x)$.

(b) [40%] Let $p = 5$. If $f(x)$ is the odd $2p$-periodic extension to $(-\infty, \infty)$ of the function $f_0(x) = 100x\,e^{10x}$ on $0 \le x \le p$, then find $f(11.3)$. The answer is not to be simplified or evaluated to a decimal.

(c) [30%] Mark the expressions which are periodic with letter **P**, those odd with **O** and those even with **E**.

$$\sin(\cos(2x)) \quad \ln|2+\sin(x)| \quad \sin(2x)\cos(x) \quad \frac{1+\sin(x)}{2+\cos(x)}$$

Answer:

(a) $f(x) = \sin(x + 2x)$ by a trig identity. Then period $= 2\pi/3$.
(b) $f(11.3) = f(11.3 - p - p) = f(1.3) = f_0(1.3) = 130e^{13}$.
(c) All are periodic of period 2π, satisfying $f(x + 2\pi) = f(x)$. The first is even and the third is odd.

3. (Fourier Series)

Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's phenomenon?

Answer:

(a) Because $f_0(x)$ is even, then $f(x)$ is even. Then the coefficient of $\sin(67\pi x)$ is zero, without computation, because all sine terms in the Fourier series of f have zero coefficient. The coefficient of $\cos(n\pi x/2)$ for $n > 0$ is given by the formula

$$a_n = \frac{1}{2}\int_{-2}^{2} f_0(x)\cos(n\pi x/2)dx = \int_0^2 x\cos(n\pi x/2)dx.$$

For $\cos(2\pi x)$, we select $n\pi x/2 = 2\pi x$, or index $n = 4$.

(b) There are no jump discontinuities, f is continous, so no Gibbs overshoot.

4. (Cosine and Sine Series)

Find the first nonzero term in the sine series expansion of $f(x)$, formed as the odd 2π-periodic extension of the function $\sin(x)\cos(x)$ on $0 < x < \pi$. Leave the Fourier coefficient in integral form, unevaluated, unless you can compute the value in a minute or two.

Answer:

Because $\sin(x)\cos(x) = (1/2)\sin(2x)$ is odd and 2π-periodic, this is the Fourier series of f. This term is for coefficient b_2, so $b_2 = 1/2$ is the first nonzero Fourier coefficient. The first nonzero term is $(1/2)\sin(2x)$.

5. (Convergence of Fourier Series)

(a) [30%] Dirichlet's kernel formula can be used to evaluate the sum $\cos(2x) + \cos(4x) + \cos(6x) + \cos(8x)$. Report its value according to that formula.

(b) [40%] The Fourier Convergence Theorem for piecewise smooth functions applies to continuously differentiable functions of period $2p$. State the theorem for this special case, by translating the results when f is smooth and the interval $-\pi \le x \le \pi$ is replaced by $-p \le x \le p$.

(c) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's overshoot at the integers $x = 0, \pm 2, \pm 4, \ldots$, (all $\pm 2n$) and nowhere else.

Answer:

(a) $\frac{1}{2} + \cos(x) + \cdots + \cos(nx) = \frac{\sin(nx + x/2)}{2\sin(x/2)}$ is used with x replaced by $2x$ and $n = 4$ to obtain the answer $0.5\sin(8x + x)/\sin(x) - 0.5$.

(b) Let f be a p-periodic smooth function on $(-\infty, \infty)$. Then for all values of x,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\pi x/p) + b_n\sin(n\pi x/p),$$

where the Fourier coefficients a_0, a_n, b_n are given by the Euler formulas:

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)dx, \quad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos(n\pi x/p)dx,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin(n\pi x/p)dx.$$

(c) Any 2-periodic continuous function f will work, if we alter the values of f at the desired points to produce a jump discontinuity. For example, define $f(x) = \sin(\pi x)$ except at the points $\pm 2n$, where $f(x) = 2$ ($f(2n) = 2$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$).