Name ______________________________

Math 3150 Problems
Chapter 4

Due date: See the internet due date. Problems are collected once a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one package per problem. Label each problem with its corresponding problem number, e.g., $\boxed{\text{Prob3.1-4}}$ or $\boxed{\text{Xc1.2-4}}$. Kindly label extra credit problems with label $\boxed{\textbf{Extra Credit}}$. You may attach this printed sheet to simplify your work.

Labeling. The label $\boxed{\text{Probx.y-z}}$ means the problem is for chapter $\boxed{\text{x}}$, section $\boxed{\text{y}}$, problem $\boxed{\text{z}}$. When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., $\boxed{\text{Xc1.0-4}}$, $\boxed{\text{Xc1.1-2}}$.

Chapter 4: 4.1-4.2 – Laplacian and Symmetric Drumhead Vibration

Prob4.1-5. (Laplacian in Spherical Coordinates)

Represent $u(x, y, z) = \left(x^2 + y^2 + z^2\right)^{3/2}$ in spherical coordinates $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$ and then decide if u satisfies Laplace's equation in spherical coordinates,

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\phi\phi}\right) = 0.$$

Xc4.1-9. (Spherical Laplacian Symmetric Case)

Supply details when $u(r, \theta, \phi)$ is independent of θ and ϕ to verify that Laplace's equation

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\phi\phi}\right) = 0$$

reduces to the simpler symmetric case equation

$$u_{rr} + \frac{2}{r}u_r = 0.$$

Prob4.0-1. (Power Series Method)

Solve $y'' + y = x + 1$, $y(0) = 0$, $y'(0) = 1$ by the power series method to obtain a series solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Answer: $y(x) = x + 1 - \cos x = x + 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$. To justify your answer, which may have a different form, compare the first five nonzero terms of the two series answers. REF: Example 3, appendix A.5, page A44

Prob4.0-2. (Euler Differential Equation)

The transformation pair $x = e^t$, $y(x) = u(t)$ changes the Euler differential equation $Ax^2y'' + Bxy' + Cy = 0$ into the constant-coefficient equation

$$A\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right) + B\frac{du}{dt} + Cu = 0,$$

with corresponding characteristic equation $Ar(r-1) + Br + C = 0$. REF: Example 4, appendix A.3, page A24

(a) Solve $x^2y'' + 4xy' + 2y = 0$.
(c) Solve $x^2y'' + xy' + 4y = 0$.

(b) Solve $2x^2y'' + 6xy' + 2y = 0$.
(d) Solve $x^2y'' - xy' + 5y = 0$.

Prob4.0-3. **(Frobenius Method)**

Solve by the Frobenius method for $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $x > 0$, where r is the largest root of the indicial equation. Find only the first three nonzero terms of the Frobenius series. Check the answer using a computer algebra system.

REF: Example 2, appendix A.6, page A55

(a) $4xy'' + 6y' + y = 0$, $4r(r-1) + 6r + 0 = 0$

(b) $4xy'' + 2y' + y = 0$ $4r(r-1) + 2r + 0 = 0$

(c) $2y'' - \frac{1}{x}y' + \frac{2}{x}y = 0$, $2r(r-1) - r + 0 = 0$

Prob4.0-4. **(Frobenius Method Case 3)**

Solve for two independent solutions of $xy'' - (2+x)y' + 2y = 0$ using the method of Frobenius and optionally a computer algebra system.

REF: Example 4, appendix A.6, page A59.

DETAILS: Indicial equation $r(r-1) - 2r + 0 = 0$, with roots $r = 0$, $r = 3$. Then $y_1(x) = x^3 \sum_{n=0}^{\infty} a_m x^m$ and $y_2(x) = ky_1(x)\ln|x| + x^0 \sum_{n=0}^{\infty} b_n x^n$.

MAPLE: There is no log term. The coefficient of $\_C1$ is $y_1(x) = 1 + x/4 + x^2/20 + \cdots$ and the coefficient of $\_C2$ is $y_2(x) = 12 + 12x + 6x^2 + \cdots$.

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de:=x^2*diff(y(x),x,x)-x*(2+x)*diff(y(x),x)+2*x*y(x)=0;
dsolve({de},y(x),series);
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References: Asmar *PDE and BVP*, Appendix A.5 or A.6, and Edwards-Penney *DE and BVP*, section 8.4, or *DE and Linear Algebra*, chapter 11.

Prob4.2-1. **(Radially Symmetric Drumhead)**

Solve the radially symmetric drumhead problem for $u(r,t)$ on the domain $0 < r < 2$, $t > 0$:

$$\begin{aligned} u_{tt}(r,t) &= u_{rr}(r,t) + \frac{1}{r}u_r(r,t), \\ u(2,t) &= 0, \\ u(r,0) &= 0, \\ u_t(r,0) &= 1. \end{aligned}$$

Xc4.2-12a. **(Series Identity for $J_0(x)$)**

Using Bessel's equation of order zero,

$$x^2y'' + xy' + x^2y = 0,$$

derive from the Frobenius method the series formula

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n}.$$

Observe that this is a special solution obtained from the Frobenius series $y = \sum_{n=0}^{\infty} a_n x^n$ by taking $a_0 = 1$.

Xc4.2-12b. **(Bessel Function Identities)**

Establish the identities

$$\begin{aligned} \int J_1(x)dx &= -J_0(x) + c, \\ \int xJ_0(x)dx &= xJ_1(x) + c. \end{aligned}$$

Chapter 4: 4.3-4.4 – Non-Symmetric Drumhead and 2D Wave Equation

Prob4.3-3. **(Non-Symmetric Drumhead)**

Solve the drumhead problem for $u(r,\theta,t)$. Plot the drumhead for $t = 0, 1, 3$ using a truncated series of four nonzero numerically approximated coefficients. Assume in the problem statement $0 < r < 2$, $0 < \theta < 2\pi$, $t > 0$.

$$\begin{aligned}
u_{tt}(r,\theta,t) &= u_{rr}(r,\theta,t) + \frac{1}{r}u_r(r,\theta,t) + \frac{1}{r^2}u_{\theta\theta}(r,\theta,t),\\
u(2,\theta,t) &= 0,\\
u(r,0,t) &= u(r,2\pi,t),\\
u_\theta(r,0,t) &= u_\theta(r,2\pi,t),\\
u(r,\theta,0) &= (4-r^2)r\sin\theta,\\
u_t(r,\theta,0) &= 1.
\end{aligned}$$

Xc4.3-13. **(Two-Dimensional Heat Conduction)**

Solve the heat conduction problem in a circular plate for $u(r,\theta,t)$. Assume in the problem statement $0 < r < 1$, $0 < \theta < 2\pi$, $t > 0$.

$$\begin{aligned}
u_t(r,\theta,t) &= u_{rr}(r,\theta,t) + \frac{1}{r}u_r(r,\theta,t) + \frac{1}{r^2}u_{\theta\theta}(r,\theta,t),\\
u(1,\theta,t) &= \sin 3\theta,\\
u(r,\theta,0) &= 0.
\end{aligned}$$

Chapter 4: 4.4 – Steady-State Temperature in a Disk

Prob4.4-5. **(Dirichlet Problem on a Disk)**

Solve the unit disk steady-state heat problem on $0 < r < 1$, $0 \le \theta < 2\pi$ for $u(r,\theta)$:

$$\begin{aligned}
u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) &= 0,\\
u(1,\theta) &= \begin{cases} 100 & 0 \le \theta \le 0.25\pi,\\ 0 & 0.25\pi < \theta < 2\pi.\end{cases}
\end{aligned}$$

Prob4.4-15a. **(Exterior Dirichlet Problem on a Disk)**

Solve the unit disk exterior steady-state heat problem on $r > 1$, $0 \le \theta < 2\pi$ for $u(r,\theta)$:

$$\begin{aligned}
u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) &= 0,\\
u(1,\theta) &= \begin{cases} 100 & 0 \le \theta \le 0.25\pi,\\ 0 & 0.25\pi < \theta < 2\pi.\end{cases}
\end{aligned}$$

Xc4.4-11. **(Dirichlet Series Formula)**

Establish the identity

$$\sum_{n=1}^{\infty} \frac{u^n}{n}\sin(n\theta) = \arctan\left(\frac{u\sin\theta}{1-u\cos\theta}\right).$$

Hint: Use the Taylor expansion (1) $\sum_{n=1}^{\infty}\frac{z^n}{n} = -ln(1-z)$. Replace u by r to derive the identity. Let $z = re^{i\theta}$ and expand the Taylor series into real and complex series. Then the imaginary part of equation (1) is the equality desired. Decoding the RHS involves writing $1 - z = (1-x) + (-y)i = r_1 e^{i\theta_1}$ followed by using the formula $-\tan(\theta_1) = -(-y)/(1-x)$.

Prob4.4-15b. **(Cartesian Coordinates)**

Use the *Dirichlet series formula* with $u = 1/r$ to convert to rectangular xy-coordinates the exterior Dirichlet problem solution in polar form

$$u(r,\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{r^{-n}(1-(-1)^n)}{n}\sin(n\theta).$$

Reference: Asmar *PDE and BVP*, section 4.4, where you will find several useful trigonometric identities. Use $(-1)^n = \cos(n\pi)$, $x = r\cos\theta$, $y = r\sin\theta$ for the conversion.

Answer: $u(r,\theta) = \frac{25}{2} + \frac{100}{\pi}(\arctan(v_1) + \arctan(v_2))$ where $v_1 = \dfrac{\sin\theta}{r-\cos\theta}$ and $v_2 = \dfrac{\sin\theta}{r+\cos\theta}$. In terms of x and y, $v_1 = \dfrac{y}{x^2+y^2-x}$, $v_2 = \dfrac{y}{x^2+y^2+x}$.

Prob4.4-15c. **(Isotherms)**

Consider the exterior Dirichlet problem on the unit disk, $r > 1$, $0 \le \theta < 2\pi$,

$$\begin{aligned} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) &= 0, \\ u(1,\theta) &= \begin{cases} 100 & 0 \le \theta \le 0.25\pi, \\ 0 & 0.25\pi < \theta < 2\pi. \end{cases} \end{aligned}$$

The isotherms are the xy-plane curves of constant temperature described by $u(x,y) = T$, where $0 \le T \le 100$. They are circles. Find their equations and plot a representative set of isotherms.

Chapter 4: 4.5 – Steady-State Temperature in a Cylinder

Prob4.5-1. **(Steady-State Temperature in a Cylinder)**

The radially symmetric case of a Dirchlet problem on a cylinder uses Laplace's equation in cylindrical coordinates. Solve the boundary value problem.

$$\begin{aligned} &\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{\partial^2 u}{\partial z^2} = 0, \\ &u(\rho,z) = 0 \text{ on the bottom } (z=0), \\ &u(\rho,z) = 0 \text{ on lateral surface } (\rho = 1,\ 0 < z < 2), \\ &u(\rho,z) = 100 \text{ on the top } z = 2,\ 0 < \rho < 1. \end{aligned}$$

Answer: $u(\rho,z) = 200\sum_{n=1}^{\infty} A_n J_0(\lambda_n\rho)\sinh(\lambda_n z)$ where λ_n is the nth positive root x of $J_0(x) = 0$. The coefficient A_n equals the reciprocal of $\lambda_n J_1(\lambda_n)\sinh(2\lambda_n)$. Suggestion: use the formulas developed in section 4.5 of Asmar.

Chapter 4: 4.6 – Poisson's Equation on a Disk

Prob4.6-5. **(Poisson's Equation on a Disk)**

Solve the Poisson disk problem by decompsition into two problems: (1) Poisson problem with zero boundary conditions, and (2) Dirichlet problem with non-homogenous boundary condition.

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -u + 1, \\ &u(1,\theta) = 0, \quad 0 \le \theta \le 2\pi. \end{aligned}$$

Answer: The problem is first converted to polar coordinates (r,θ), using $u(x,y) = u(r,\theta)$ and $x = r\cos\theta, y = r\sin\theta$. Then $u(r,\theta) = 2\sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$ where λ_n is the nth positive root x of $J_0(x) = 0$. The coefficient A_n equals the reciprocal of $\lambda_n(1-\lambda_n^2)J_1(\lambda_n)$. Suggestion: use the formulas developed in sections 4.4 and 4.6 of Asmar.