Chapter 3: 3.1 – Examples in Physics and Engineering

Prob3.1-3. (Classification)
Classify \( u_{xx} - u_t = 2u \) as linear, nonlinear, homogeneous, non-homogeneous, and report the order of the equation.

Prob3.1-7. (Laplace Equation)
Verify that \( u(x, y) = e^y \cos x + x + y \) is a solution of Laplace’s partial differential equation.

Chapter 3: 3.2-3.3 – One Dimensional Wave Equation

Prob3.2-1. (Wave Equation)
Derive the equation \( u_{tt} = 10^5 u_{xx} \) for the vibrations of a stretched homogeneous string with linear density \( \rho = 0.001 \) kg/m and tension \( \tau = 100 \) N, with no forces other than the tension. State all assumptions used to obtain the model. Make the presentation brief, by referencing a textbook for derivation details and results.

Prob3.3-9a. (Separation of Variables)
Solve \( u_{tt} = u_{xx}, \; u(0,t) = u(1,t) = 0, \; u(x,0) = x(1-x), \; u_t(x,0) = \sin \pi x, \; t \geq 0, \; 0 \leq x \leq 1. \) The model is for a guitar string of unit length.

Prob3.3-9b. (Filmstrip Plots)
Plot partial sums of the answer to the previous problem,
\[
u(x,t) = \frac{1}{\pi} \sin(\pi x) \sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3} \sin(2m\pi x + \pi x) \cos(2m\pi t + \pi t),
\]
at \( t = 0, 1, 2, 3. \) Choose the number of series terms for the four graphics by making the first graphic match \( x(1-x) \) on \( 0 \leq x \leq 1. \) This filmstrip has 4 frames, each frame corresponding to a time \( t. \) A frame has graph window \( 0 \leq x \leq 1, \; a \leq u \leq b \) (you must choose \( a, b \)).

Prob3.3-9c. (Surface Plot)
Plot a specific partial sum of the answer
\[
u(x,t) = \frac{1}{\pi} \sin(\pi x) \sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3} \sin(2m\pi x + \pi x) \cos(2m\pi t + \pi t)
\]
on the domain \( 0 \leq x \leq 1, \; 0 \leq t \leq 4. \) Use all features possible of the 3D graphics program in order to produce the best plot with fine accuracy, view and colors.
Prob3.3-13. (Damped Vibrations of a String)
Solve the problem
\[ \begin{aligned}
  u_{tt}(x, t) + u_t(x, t) &= u_{xx}(x, t), \\
  u(0, t) &= 0, \\
  u(\pi, t) &= 0, \\
  u(x, 0) &= \sin x, \\
  u_t(x, 0) &= 0.
\end{aligned} \]

Chapter 3: 3.4 – d’Alembert’s Method

Prob3.4-15. (d’Alembert’s Solution)
Consider the problem
\[ \begin{aligned}
  u_{tt} &= u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0, \\
  u(0, t) &= 0, \\
  u(1, t) &= 0, \\
  u(x, 0) &= f(x), \\
  u_t(x, 0) &= 0.
\end{aligned} \]
Assume \( f(x) = 4x \) on \( 0 \leq x \leq 0.25 \), \( f(x) = 2 - 4x \) on \( 0.25 < x \leq 0.5 \), \( f(x) = 0 \) on \( 0.5 < x \leq 1 \).

(a) Find a solution formula for \( u(x, t) \) using d’Alembert’s method.
(b) Plot a 3-frame filmstrip of the string shape at times \( t = 0, 0.25, 0.5 \).

# EXAMPLE. Let \( f(x)=4x \) on \( [0,.25] \), \( f(x)=2-4x \) on \( [.25,.5] \), \( f(x)=0 \) otherwise
# Asmar 3.4-15, D’Alembert’s solution of the wave equation, f=pulses,g=0
pulse:=(x,a,b)->piecewise(x<a,0,x<b,1,0);
f:=x->4*x*pulse(x,0,1/4)+(2-4*x)*pulse(x,1/4,1/2); #plot(f(x),x=0..1);
F0:=x->piecewise(x<0,-f(-x),f(x)); # Odd extension of f(x)
plot(F0(x),x=-1..1);
F:=x->F0(x-2*floor(x/2+1/2)) # plot(F(x),x=-2..3);
u:=(x,t)->(1/2)*(F(x+t)+F(x-t)); #plot(u(x,0.7),x=-2..2);
plots[animate]( plot, [u(x,t),x=-3..3], t=0..1.5, trace=0, frames=50 );

See the web site links for updates to this sample maple code.

Xc3.4-18. (Energy Conservation and d’Alembert’s Solution)
Define
\[ E(t) = \frac{1}{2} \int_0^L \left( u_t^2(x, t) + c^2 u^2_x(x, t) \right) dx. \]
Prove the energy conservation law, which says that the energy during free vibrations of a string is constant for all time.
Hint: Show \( dE/dt = 0 \).

Chapter 3: 3.5-3.6 – One Dimensional Heat Equation

Prob3.5-13. (Nonhomogeneous Heat Equation)
Consider the one-dimensional heat conduction problem
\[ \begin{aligned}
  u_t &= u_{xx}, \quad 0 \leq x \leq \pi, \quad t > 0, \\
  u(0, t) &= 0, \\
  u(\pi, t) &= 0, \\
  u(x, 0) &= f(x).
\end{aligned} \]
Assume \( f(x) = 33x \) on \( 0 < x < \pi/2 \), \( f(x) = 33\pi - 33x \) on \( \pi/2 < x < \pi \). Find a solution formula for the temperature \( u(x, t) \).
Prob3.6-3. (Heat Conduction in an Insulated Bar)
Consider the one-dimensional heat conduction problem
\[ u_t = u_{xx}, \ 0 \leq x \leq 1, \ t > 0, \]
\[ u_x(0, t) = 0, \]
\[ u_x(1, t) = 0, \]
\[ u(x, 0) = \cos \pi x \]
Find a solution formula for the temperature \( u(x, t) \) at location \( x \) along the bar at time \( t \). Hint: Don’t integrate!

Remark. The book’s problem 3.6-3 has a piecewise example, using \( u(x, 0) = f(x) \). See the maple advice for problem 3.5-13, to handle that case.

Chapter 3: 3.7 – Two Dimensional Equations

Prob3.7-5a. (Vibrations of a Membrane)
Consider the rectangular drumhead problem, in which we assume \( 0 < x < 1, 0 < y < 1, t > 0 \):
\[ u_{tt}(x, y, t) = \frac{1}{\pi^2} (u_{xx}(x, y, t) + u_{yy}(x, y, t)), \]
\[ u(0, y, t) = 0, \]
\[ u(1, y, t) = 0, \]
\[ u(x, 0, t) = 0, \]
\[ u(x, 1, t) = 0, \]
\[ u(x, y, 0) = 0, \]
\[ u_t(x, y, 0) = 1. \]
Solve for the drumhead deflection \( u(x, y, t) \).

Prob3.7-5b. (Membrane Snapshots)
Consider the solution of the rectangular drumhead problem given by the series
\[ u(x, y, t) = \frac{16}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{nm\sqrt{n^2 + m^2}} \sin(m\pi x) \sin(n\pi y) \sin \left( t\sqrt{m^2 + n^2} \right). \]
Illustrate the various shapes of the drumhead during vibration, by plotting suitable surface snapshots at times \( t = 1, 2, 3 \). The snapshot at \( t = 0 \) should be the initial flat membrane shape \( u = 0 \). Choose suitable partial sums to reveal adequate detail in the plots.

Prob3.7-12. (Heat Conduction in a Plate)
Consider the rectangular plate heat conduction problem in which we assume \( 0 \leq x \leq 1, 0 < y < 1, t > 0 \):
\[ u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t), \]
\[ u(0, y, t) = 0, \]
\[ u(1, y, t) = 0, \]
\[ u(x, 0, t) = 0, \]
\[ u(x, 1, t) = 0, \]
\[ u(x, y, 0) = x(1-x)y(1-y). \]
Solve for the plate temperature \( u(x, y, t) \).

Xc3.7-12. (Heat Conduction in a Plate)
Find a series estimate for the solution \( u(x, y, t) \) of the rectangular plate heat conduction problem which shows that \( |u(x, y, t)| \leq Me^{-\alpha t} \) for some number \( M > 0 \) and some constant \( \alpha > 0 \). Then conclude that
\[ \lim_{t \to \infty} u(x, y, t) = 0, \]
which implies the plate temperature \( u \) stabilizes to the edge temperature \( u = 0 \) as \( t \) approaches infinity.
Problem notes.

The Cauchy-Schwartz inequality is used to find an upper estimate of $|u|^2$ as a product of two positive series. One series is numeric, and Bessel’s inequality can be used to determine an upper bound $M_1$ for it. The other series in the product is a series of functions, each function an exponential function bounded above by $e^{-\beta t}$, where $\beta > 0$ is a fixed constant. A clever analysis of the exponential factors, using the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, shows that the second series is bounded by some constant $M_2 > 0$ times $e^{-\rho t}$, where $\rho = \beta/2$. Taking square roots across $|u|^2 \leq M_1 M_2 e^{-\rho t}$ implies that constants $M = \sqrt{M_1 M_2} > 0$ and $\alpha = \rho/2 > 0$ satisfy $|u| \leq M e^{-\alpha t}$.

Chapter 3: 3.8-3.9 – Laplace’s and Poisson’s Equations

Prob3.8-2. (Steady-State Temperature in a Plate)
Consider the rectangular plate steady-state heat conduction problem in which we assume the plate is given by $0 \leq x \leq 1$, $0 < y < 1$:

\[
\begin{align*}
    u_{xx}(x,y) + u_{yy}(x,y) &= 0, \\
    u(x,0) &= 0, \\
    u(x,1) &= 100, \\
    u(0,y) &= 0, \\
    u(1,y) &= 100.
\end{align*}
\]

(a) Draw a figure for the Dirichlet problem, showing the edge temperatures on the plate. Break the problem into two subproblems, decomposing $u = u_1 + u_2$. Draw figures for each subproblem.

(b) Solve for the temperatures $u_1(x,y)$ and $u_2(x,y)$.

(c) Report the solution to the original problem, $u = u_1 + u_2$.

Notes. In the general problem of Nakhle’s section 3.8, $f_1(x) = g_1(y) = 0$ and $f_2(x) = g_2(y) = 100$. In the summary shaded display before the 3.8 exercises, $A_n = C_n = 0$ and $B_n$, $D_n$ are computed from equations (5), (6). Example 2 in Nakhle’s section 3.8 solves for $B_n$, therefore you have an easy answer check for half the problem.

Prob3.9-3. (Poisson Problem)
Consider the rectangular plate steady-state Poisson heat conduction problem in which we assume the plate is given by $0 \leq x \leq 1$, $0 < y < 1$:

\[
\begin{align*}
    u_{xx}(x,y) + u_{yy}(x,y) &= \sin \pi x, \\
    u(x,0) &= 0, \\
    u(x,1) &= x, \\
    u(0,y) &= 0, \\
    u(1,y) &= 0.
\end{align*}
\]

(a) Draw a figure for the Poisson problem with zero boundary conditions [see (b)]. Draw a second figure for the corresponding Dirichlet problem with identical boundary conditions [see (c)].

(b) Solve for the temperature $u_1(x,y)$ satisfying the Poisson problem

\[
\begin{align*}
    u_{xx}(x,y) + u_{yy}(x,y) &= \sin \pi x, \\
    u(x,0) &= 0, \\
    u(x,1) &= 0, \\
    u(0,y) &= 0, \\
    u(1,y) &= 0.
\end{align*}
\]

(c) Solve for the temperature $u_2(x,y)$ satisfying the Dirichlet problem

\[
\begin{align*}
    u_{xx}(x,y) + u_{yy}(x,y) &= 0, \\
    u(x,0) &= 0, \\
    u(x,1) &= x, \\
    u(0,y) &= 0, \\
    u(1,y) &= 0.
\end{align*}
\]
(d) Report the solution to the original Poisson problem, which is $u = u_1 + u_2$.

Notes. Problem (c) is solved in section 3.8 of Nakhle’s textbook, with summary in the shaded display just before the exercises 3.8. In this display, $A_n = C_n = D_n = 0$ and $B_n$ must be computed from (5) using $f_2(x) = x$. Problem (b) is solved from Nakhle’s section 3.9 equations (2) and (4). A similar problem is solved in Example 1. The challenge is the double integration in (4) with $f(x, y) = \sin \pi x$. Luckily, this is an iterated double integral, evaluated by two successive one-variable integrations:

$$E_{mn} = \frac{-4}{\lambda_{mn}} \int_0^1 \sin \pi x \sin m\pi x dx \int_0^1 \sin n\pi y dy.$$