Thus this formula is an alternative way of writing the series solution (4).
(a) Substitute the coefficients (5) in the series solution (4) and obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\Big\{1 + 2\sum_{n=1}^{\infty} \cos[n(\theta-\phi)]\left(\frac{r}{a}\right)^n\Big\}\,d\phi.$$

[Hint: Before you do the substitution, replace the dummy variable θ in (5) by ϕ.]
(b) Derive the Poisson formula using (a) and Exercise 28.

4.5 Laplace's Equation in a Cylinder

In this section we treat certain radially symmetric Dirichlet problems in cylindrical regions. In cylindrical coordinates, Laplace's equation, with no ϕ dependence, is

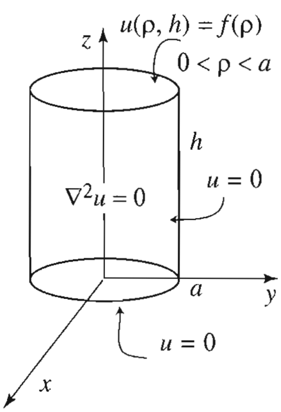

Figure 1

$$(1)\qquad \nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(See (4), Section 4.1.) The first problem that we will consider models the steady-state temperature distribution inside a cylinder with lateral surface and bottom kept at zero temperature and with radially symmetric temperature distribution at the top, as shown in Figure 1.

DIRICHLET PROBLEM IN A CYLINDER WITH ZERO LATERAL TEMPERATURE

The solution of Laplace's equation (1) with boundary conditions

$$u(\rho,0) = 0, \quad 0<\rho<a,$$
$$u(a,z) = 0, \quad 0<z<h,$$
$$u(\rho,h) = f(\rho), \quad 0<\rho<a,$$

is

$$(2)\qquad u(\rho,z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho)\sinh \lambda_n z,$$

where

$$(3)\qquad A_n = \frac{2}{\sinh(\lambda_n h)a^2 J_1^2(\alpha_n)}\int_0^a f(\rho)J_0(\lambda_n\rho)\rho\,d\rho, \quad \lambda_n = \frac{\alpha_n}{a},$$

and α_n is the nth positive zero of J_0, the Bessel function of order 0.

Proof Using the method of separation of variables and setting $u(\rho,z) = R(\rho)Z(z)$, we get the equations $\rho^2 R'' + \rho R' - k\rho^2 R = 0$, $R(a) = 0$, and $Z'' + kZ = 0$, $Z(0) = 0$, where k is the separation constant. We also require that R be bounded at $\rho = 0$, since we are solving for the temperature inside the cylinder. If $k = 0$, it is straightforward to check that we only get the solution $R = 0$. If $k > 0$, say $k = \lambda^2$, then we get the **parametric form of the modified Bessel equation of order 0** defined in Exercise 7 (see also Exercises 29 and 30, Section 4.7). The general solution in this case is a linear combination of the **modified Bessel functions of the first** and **second kind**, I_0 and K_0, shown in Figure 2 (see Exercise 7). Since the first one is positive and strictly increasing for $\rho > 0$, and the second one is unbounded near zero, we conclude that no nontrivial bounded linear combination of these functions can satisfy the boundary conditions on R. So this leaves the only possibility $k = -\lambda^2 < 0$. In this case we have

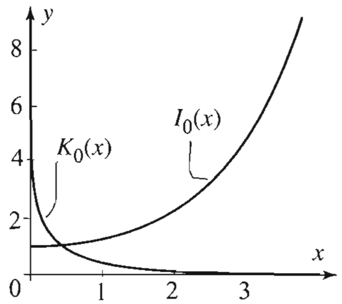

Figure 2 Modified Bessel functions.

$$\rho^2 R'' + \rho R' + \lambda^2 \rho^2 R = 0, \quad R(a) = 0,$$
$$Z'' - \lambda^2 Z = 0, \quad Z(0) = 0.$$

Applying Theorem 3, Section 4.8, we find that $R = R_n(\rho) = J_0(\lambda_n \rho)$, where $\lambda_n = \alpha_n / a$, $n = 1, 2, \ldots$. Solving the equation for Z with $\lambda = \lambda_n$, we find

$$Z_n(z) = \sinh \lambda_n z \quad n = 1, 2, \ldots.$$

Superposing the product solutions we get (2) as a solution. To determine the unknown coefficients A_n, we set $z = h$ and get the Bessel series expansion

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho) \sinh \lambda_n h.$$

Thus $A_n \sinh \lambda_n h$ must be the nth Bessel coefficient of $f(\rho)$, and so (3) follows from Theorem 2, Section 4.8. ■

As a second illustration, we consider a boundary value problem with a nonzero boundary condition on the lateral surface of the cylinder (see Figure 3).

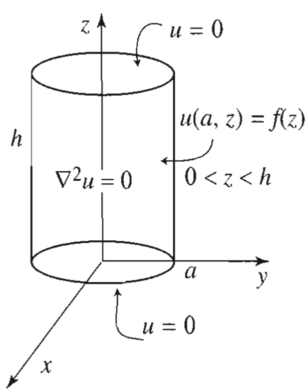

Figure 3

DIRICHLET PROBLEM IN A CYLINDER WITH NONZERO LATERAL TEMPERATURE

The solution of Laplace's equation (1) with boundary conditions

$$u(\rho, 0) = u(\rho, h) = 0, \quad 0 < \rho < a,$$
$$u(a, z) = f(z), \quad 0 < z < h,$$

is

(4)
$$u(\rho, z) = \sum_{n=1}^{\infty} B_n I_0\left(\frac{n\pi}{h}\rho\right) \sin \frac{n\pi}{h} z,$$

where I_0 is the modified Bessel function of the first kind of order 0, and

(5)
$$B_n = \frac{2}{I_0\left(\frac{n\pi a}{h}\right) h} \int_0^h f(z) \sin \frac{n\pi}{h} z \, dz.$$

The derivation of the solution is very much like the one we did previously, except that now the interesting case of the separation constant is $k = \nu^2 > 0$. The details are left to Exercise 8.

Exercises 4.5

In Exercises 1–4, find the steady-state temperature in the cylinder of Figure 1 for the given temperature distribution of its top. Take $a = 1$, and $h = 2$.

1. $f(\rho) = 100$.

2. $f(\rho) = 100 - \rho^2$.

3. $f(\rho) = \begin{cases} 100 & \text{if } 0 < \rho < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < \rho < 1. \end{cases}$

4. $f(\rho) = 70\, J_0(\rho)$.

5. (a) Find the steady-state temperature in the cylinder with boundary values as shown in Figure 4.
(b) Solve (1) for the boundary conditions

$$u(\rho, 0) = f_1(\rho), \quad 0 < \rho < a,$$
$$u(a, z) = 0, \quad 0 < z < h,$$
$$u(\rho, h) = f_2(\rho), \quad 0 < \rho < a.$$

[Hint: Combine (a) with the solution in this section.]

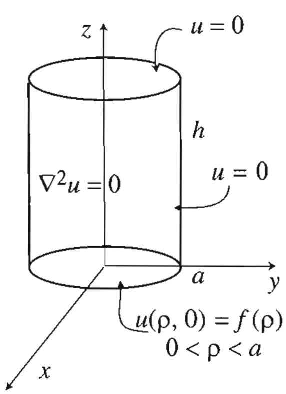

Figure 4 for Exercise 5.

6. Solve (1) for the boundary conditions

$$u(\rho, 0) = 100, \quad 0 < \rho < 1,$$
$$u(1, z) = 0, \quad 0 < z < 2,$$
$$u(\rho, 2) = 100, \quad 0 < \rho < 1.$$

7. Make the substitution $x = \lambda\rho$ ($\lambda > 0$) in the **parametric form of the modified Bessel equation** $\rho^2 R'' + \rho R' - \lambda^2\rho^2 R = 0$ and obtain that its general solution is $y = c_1\, I_0(\lambda\rho) + c_2\, K_0(\lambda\rho)$, where I_0 and K_0 are the modified Bessel functions of the first and second kind. [Hint: Use Exercises 29 and 30, Section 4.7.]

Project Problem: Lateral surface with nonzero temperature. Do Exercises 8 and 9.

8. In this exercise we derive (4) and (5).
(a) Refer to the Dirichlet problem in the cylinder with boundary conditions as given just before (4). Use the separation of variables method and obtain

$$Z'' + \nu^2 Z = 0, \quad Z(0) = 0 \text{ and } Z(h) = 0,$$
$$\rho^2 R'' + \rho R' - \nu^2\rho^2 R = 0.$$

(b) Show that the only possible solutions of the first equation correspond to $\nu_n = \frac{n\pi}{h}$ and hence are

$$Z_n(z) = \sin\frac{n\pi}{h}z, \quad n = 1, 2, \ldots.$$

(c) Derive (4) and (5). [Hint: Use Exercise 7.]

9. Solve (1) for the boundary conditions

$$u(\rho, 0) = u(\rho, 2) = 0, \quad 0 < \rho < 1,$$
$$u(1, z) = 10z, \quad 0 < z < 2.$$