4. $a=1,\ c=1,\ f(r)=0,\ g(r)=J_0(\alpha_3 r)$.
[Hint: Orthogonality relations in Section 4.8.]

5. $a=1,\ c=1,\ f(r)=J_0(\alpha_1 r),\ g(r)=0$.
[Hint: Orthogonality relations in Section 4.8.]

6. $a=2,\ c=1,\ f(r)=1-r,\ g(r)=0$.

7. $a=1,\ c=1,\ f(r)=J_0(\alpha_3 r),\ g(r)=1-r^2$.

8. $a=1,\ c=1,\ f(r)=\frac{1}{128}(3-4r^2+r^4),\ g(r)=0$.
[Hint: Integration by parts, Example 2.]

9. (a) Find the solution in Exercise 3 for an arbitrary value of $c>0$.
(b) Describe what happens to the solution as c increases.

10. Project Problem: Radially symmetric heat equation on a disk. Use the methods of this section to show that the solution of the heat boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right), &\quad 0<r<a,\ t>0,\\ u(a,t)=0, &\quad t>0,\\ u(r,0)=f(r), &\quad 0<r<a, \end{aligned}$$

is

$$u(r,t)=\sum_{n=1}^{\infty} A_n e^{-c^2\lambda_n^2 t} J_0(\lambda_n r),$$

with

$$A_n=\frac{2}{a^2 J_1^2(\alpha_n)}\int_0^a f(r)J_0(\lambda_n r)r\,dr,$$

where $\lambda_n=\frac{\alpha_n}{a}$, and $\alpha_n = n$th positive zero of J_0.

11. (a) Solve the heat problem of Exercise 10 when $f(r)=100,\ 0<r<a$. What does your solution represent?
(b) Approximate the temperature of the hottest point on the plate at time $t=3$, given that $a=1$ and $c=1$. Where is this point on the plate? Justify your answer intuitively.

12. Project Problem: Integral identities with Bessel functions.
(a) Use (7) and (8), Section 4.8, to establish the identities

$$\int J_1(x)\,dx=-J_0(x)+C \quad\text{and}\quad \int xJ_0(x)\,dx = xJ_1(x)+C\,.$$

In the rest of this problem we generalize these identities.
(b) By integrating (5), Section 4.8, show that

$$\int J_{p+1}(x)\,dx=\int J_{p-1}(x)\,dx-2J_p(x)\,.$$

(c) Use the first integral in (a), (b), and induction to establish that

$$\int J_{2n+1}(x)\,dx=-J_0(x)-2\sum_{k=1}^{n}J_{2k}(x)+C, \qquad n=0,\,1,\,2,\,\ldots.$$

As an illustration, derive the following identities:

$$\int J_3(x)\,dx = -J_0(x) - 2J_2(x) + C,$$

$$\int J_5(x)\,dx = -J_0(x) - 2J_2(x) - 2J_4(x) + C\,.$$

(d) By integrating (3), Section 4.8, show that

$$xJ_{p+1}(x) + p\int J_{p+1}(x)\,dx = \int xJ_p(x)\,dx\,.$$

[Hint: Evaluate the integral of $xJ_p'(x)$ by parts.]
(e) Take $p = 2n$ in (d) and use (c) to prove that for $n = 0,\ 1,\ 2,\ \ldots,$

$$\int xJ_{2n}(x)\,dx = xJ_{2n+1}(x) - 2nJ_0(x) - 4n\sum_{k=1}^{n} J_{2k}(x) + C\,.$$

Derive the following identities:

$$\int xJ_2(x)\,dx = xJ_3(x) - 2J_0(x) - 4J_2(x) + C,$$

$$\int xJ_4(x)\,dx = xJ_5(x) - 4J_0(x) - 8J_2(x) - 8J_4(x) + C\,.$$

4.3 Vibrations of a Circular Membrane: General Case

We continue our study of the vibrating circular membrane, now without any symmetry assumptions. We will solve the **two dimensional wave equation in polar coordinates**:

(1)
$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),}$$

where $0 < r < a,\ 0 < \theta < 2\pi,\ t > 0$. Here $u = u(r, \theta, t)$ denotes the deflection of the membrane at the point (r, θ) at time t. The **initial conditions** (displacement and velocity) are

(2)
$$u(r, \theta, 0) = f(r, \theta), \qquad \frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta),$$

$0 < r < a,\ 0 < \theta < 2\pi$. The requirement that the edges of the membrane be held fixed translates into the **boundary condition**

(3)
$$u(a, \theta, t) = 0, \quad 0 < \theta < 2\pi, t > 0\,.$$

Since θ is a polar angle, (r, θ) and $(r, \theta + 2\pi)$ represent the same point, and hence $u(r, \theta, t) = u(r, \theta + 2\pi, t)$. In other words, u is 2π-periodic in θ. Consequently,

$$u(r, 0, t) = u(r, 2\pi, t) \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r, 0, t) = \frac{\partial u}{\partial \theta}(r, 2\pi, t). \tag{4}$$

Separation of Variables

We start by deriving the general solution of (1) subject to the boundary condition (3). We use the method of separation of variables and set $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$. Differentiating u, substituting into (1), and separating variables gives

$$\frac{T''}{c^2 T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2 \Theta}.$$

The left side depends only on t and the right side only on r and θ. Therefore, each side must equal a constant k. Expecting periodic solutions in T, we take $k = -\lambda^2$. Thus

$$\frac{T''}{c^2 T} = -\lambda^2, \quad \text{and} \quad \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2 \Theta} = -\lambda^2.$$

Separating variables in the second equation we get

$$\lambda^2 r^2 + \frac{r^2 R''}{R} + \frac{rR'}{R} = \mu^2 \quad \text{and} \quad -\frac{\Theta''}{\Theta} = \mu^2.$$

We have chosen a nonnegative sign for the separating constant μ^2 because the solutions of the equation in Θ have to be 2π-periodic. The boundary condition (3) becomes $R(a)\Theta(\theta)T(t) = 0$ for $0 < \theta < 2\pi$ and $t > 0$. To avoid the trivial solution, we impose the condition $R(a) = 0$. Similarly, using (4), we find that $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. Thus we have arrived at the following separated equations:

$$\begin{gathered}\Theta'' + \mu^2 \Theta = 0, \quad \Theta(0) = \Theta(2\pi), \; \Theta'(0) = \Theta'(2\pi), \\ r^2 R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0, \quad R(a) = 0, \\ T'' + c^2 \lambda^2 T = 0.\end{gathered}$$

Solving the Separated Equations

Note that we start with the Θ equation, since we have a full complement of boundary conditions for it, and it contains only one separation constant. After determining that $\mu = m$, $m = 0, 1, 2, 3, \ldots$, we can turn to the equation in R and determine which values of the separation constant λ allow for nontrivial solutions. The T equation is dealt with last.

We begin by solving for Θ. For $\mu = 0$ the solution is a constant A_0. If $\mu \neq 0$, the general solution is of the form $\Theta(\theta) = c_1 \cos \mu\theta + c_2 \sin \mu\theta$. To satisfy the boundary conditions we must take μ to be an integer. Thus

$$\Theta_m(\theta) = A_m \cos m\theta + B_m \sin m\theta, \quad m = 0, 1, 2, \ldots .$$

(Note that negative values of m do not contribute any new solutions.) Setting $\mu = m$ in the equation for R, we get

$$r^2R'' + rR' + (\lambda^2 r^2 - m^2)R = 0, \quad R(a) = 0 .$$

This is the **parametric form of Bessel's equation of order** m which is treated in Theorem 3, Section 4.8. Quoting from this theorem, we have

We get J_m's here, and not Y_m's or a combination of J_m's and Y_m's because, on physical grounds, we insist that our solutions remain bounded at $r = 0$.

$$R(r) = R_{mn}(r) = J_m(\lambda_{mn} r), \quad m = 0, 1, 2, \ldots, \; n = 1, 2, \ldots ,$$

where $\lambda_{mn} = \alpha_{mn}/a$ and α_{mn} is the nth positive zero of the Bessel function J_m. For $\lambda = \lambda_{mn}$ the equation in T becomes $T'' + c^2\lambda_{mn}^2 T = 0$ with solutions

$$A_{mn} \cos c\lambda_{mn} t \quad \text{and} \quad B_{mn} \sin c\lambda_{mn} t .$$

Using the expressions for R, Θ, and, T, we arrive at the product solutions of (1) and (3):

$$(5) \qquad u_{mn}(r, \theta, t) = J_m(\lambda_{mn} r)(a_{mn} \cos m\theta + b_{mn} \sin m\theta) \cos c\lambda_{mn} t$$

and

$$(6) \qquad u^*_{mn}(r, \theta, t) = J_m(\lambda_{mn} r)(a^*_{mn} \cos m\theta + b^*_{mn} \sin m\theta) \sin c\lambda_{mn} t,$$

where $m = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$. Note that we have replaced the coefficient $A_m A_{mn}$ by a_{mn}, and similarly for b_{mn}, a^*_{mn}, and b^*_{mn}. While this may appear to be just relabeling of the unknown coefficients, in fact, it provides a more convenient choice of solutions that will be needed as we proceed. Note too that b_{0n} and b^*_{0n} will never be needed, since $\sin m\theta = 0$ when $m = 0$, and so for the sake of definiteness we take them to be 0.

Superposition Principle and the General Solution

The superposition principle suggests adding all the functions in (5) and (6). The resulting sum is displayed in (16) below. Because of the complexity of this solution, we consider two cases separately: one in which the initial velocity g is zero, and a second in which the initial displacement f is zero. The general solution is then obtained by combining these two cases.

EXAMPLE 1 Vibrations of a membrane with zero initial velocity

Solve the boundary value problem consisting of (1)–(3) given that $g = 0$.

Solution The initial conditions in this case are

$$u(r,\theta,0) = f(r,\theta), \quad \frac{\partial u}{\partial t}(r,\theta,0) = 0, \quad 0 < r < a, \ 0 < \theta < 2\pi .$$

It is easily seen that the only product solutions that meet the second condition are those given by (5). Thus the superposition principle suggests a solution of the form

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\,(a_{mn}\cos m\theta + b_{mn}\sin m\theta)\cos c\lambda_{mn} t\,. \tag{7}$$

Setting $t = 0$, we get

$$\boxed{f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(a_{mn}\cos m\theta + b_{mn}\sin m\theta)\,.} \tag{8}$$

This surely is a sort of a generalized Fourier series of $f(r,\theta)$ in terms of the functions $J_m(\lambda_{mn} r)\cos m\theta$ and $J_m(\lambda_{mn} r)\sin m\theta$, and hence a_{mn} and b_{mn} are the corresponding generalized Fourier coefficients of the function f. This fact and many important related applications are explored in Section 4.6 (see in particular Theorems 1 and 2 of that section). We now proceed to determine a_{mn} and b_{mn}, using properties of the usual Fourier series and Bessel series.

Fix r and think of $f(r,\theta)$ as a (2π-periodic) function of θ. To facilitate the use of Fourier series, we write (8) as

$$\begin{aligned} f(r,\theta) &= \overbrace{\sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)}^{=a_0(r)} + \sum_{m=1}^{\infty}\Bigg\{\Bigg(\overbrace{\sum_{n=1}^{\infty} a_{mn} J_m(\lambda_{mn} r)}^{=a_m(r)}\Bigg)\cos m\theta \\ &\qquad + \Bigg(\overbrace{\sum_{n=1}^{\infty} b_{mn} J_m(\lambda_{mn} r)}^{=b_m(r)}\Bigg)\sin m\theta\Bigg\} \\ &= a_0(r) + \sum_{m=1}^{\infty}\left(a_m(r)\cos m\theta + b_m(r)\sin m\theta\right)\,. \end{aligned}$$

Now we see clearly that (for fixed r) $a_0(r)$, $a_m(r)$, and $b_m(r)$ are the Fourier coefficients in the Fourier series expansion of $\theta \mapsto f(r,\theta)$. Using (2)–(4), Section 2.2, we conclude that

$$a_0(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,d\theta = \sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r), \tag{9}$$

$$a_m(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos m\theta\,d\theta = \sum_{n=1}^{\infty} a_{mn} J_m(\lambda_{mn} r), \tag{10}$$

$$b_m(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin m\theta\,d\theta = \sum_{n=1}^{\infty} b_{mn} J_m(\lambda_{mn} r), \tag{11}$$

for $m = 1, 2, \ldots$ Now let r vary and think of the last three series as the Bessel series expansions of order $m = 0, 1, 2, \ldots$ of the functions $a_0(r)$, $a_m(r)$, and $b_m(r)$, respectively. The coefficients in these series are Bessel coefficients and so from (17), Section 4.8, we obtain

$$a_{0n} = \frac{2}{a^2 J_1^2(\alpha_{0n})}\int_0^a a_0(r) J_0(\lambda_{0n} r) r\,dr,$$

$$a_{mn} = \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a a_m(r) J_m(\lambda_{mn} r) r\,dr,$$

$$b_{mn} = \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a b_m(r) J_m(\lambda_{mn} r) r\,dr.$$

Now using (9)–(11), we get

$$a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})}\int_0^a\int_0^{2\pi} f(r,\theta) J_0(\lambda_{0n} r) r\,d\theta\,dr, \tag{12}$$

$$a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\cos m\theta\, J_m(\lambda_{mn} r) r\,d\theta\,dr, \tag{13}$$

$$b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\sin m\theta\, J_m(\lambda_{mn} r) r\,d\theta\,dr, \tag{14}$$

for $m,\ n = 1, 2 \ldots$. Substituting these coefficients in (7) completes the solution of the problem. ■

Before giving a numerical application, we present a useful identity involving Bessel functions.

A USEFUL IDENTITY

For any $k \geq 0$, $a > 0$, and $\alpha > 0$, we have

$$\int_0^a (a^2 - r^2) r^{k+1} J_k\left(\frac{\alpha}{a} r\right) dr = 2\,\frac{a^{k+4}}{\alpha^2}\, J_{k+2}(\alpha). \tag{15}$$

Proof We first make a change of variables, $\frac{\alpha}{a} r = x$, $dr = \frac{a}{\alpha}\,dx$, and transform the integral into

$$\frac{a^{k+2}}{\alpha^{k+2}}\int_0^\alpha \left(a^2 - \frac{a^2 x^2}{\alpha^2}\right) x^{k+1} J_k(x)\,dx$$
$$= \frac{a^{k+4}}{\alpha^{k+2}}\int_0^\alpha x^{k+1} J_k(x)\,dx - \frac{a^{k+4}}{\alpha^{k+4}}\int_0^\alpha x^{k+3} J_k(x)\,dx.$$

From (7), Section 4.8, with $p = k$, the first term is

$$\frac{a^{k+4}}{\alpha^{k+2}} \left[x^{k+1} J_{k+1}(x)\right]_0^\alpha = \frac{a^{k+4}}{\alpha} J_{k+1}(\alpha).$$

The second integral can be evaluated with the help of (7), Section 4.8, and integration by parts. Let $u = x^2$, $dv = x^{k+1} J_k(x)\,dx$, then $du = 2x\,dx$, $v = x^{k+1} J_{k+1}(x)$. Hence the second term becomes

$$-\frac{a^{k+4}}{\alpha^{k+4}} \left[x^{k+3} J_{k+1}(x)\right]_0^\alpha + 2\frac{a^{k+4}}{\alpha^{k+4}} \int_0^\alpha x^{k+2} J_{k+1}(x)\,dx.$$

Using (7), Section 4.8, one more time and simplifying, we get

$$-\frac{a^{k+4}}{\alpha} J_{k+1}(\alpha) + 2\frac{a^{k+4}}{\alpha^2} J_{k+2}(\alpha),$$

and (15) follows. ■

EXAMPLE 2 A vibrating membrane

Refer to Example 1 and determine the solution $u(r, \theta, t)$ when $a = c = 1$, $f(r, \theta) = (1 - r^2) r \sin\theta$, $g(r, \theta) = 0$.

Solution From (12), we have

$$a_{0n} = \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^1 \int_0^{2\pi} (1 - r^2) r \sin\theta J_0(\alpha_{0n} r) r\,d\theta\,dr = 0,$$

because $\int_0^{2\pi} \sin\theta\,d\theta = 0$. A similar argument using (13), (14), and the orthogonality of the trigonometric functions shows that $a_{mn} = 0$ for all m and n, and $b_{mn} = 0$, except when $m = 1$, in which case we have

$$\begin{aligned} b_{1n} &= \frac{2}{\pi J_2^2(\alpha_{1n})} \int_0^1 \int_0^{2\pi} (1 - r^2) r \sin^2\theta\, J_1(\alpha_{1n} r) r\,d\theta\,dr \\ &= \frac{2}{J_2^2(\alpha_{1n})} \int_0^1 (1 - r^2) r^2 J_1(\alpha_{1n} r)\,dr \end{aligned}$$

because $\frac{1}{\pi} \int_0^{2\pi} \sin^2\theta\,d\theta = 1$. We now appeal to (15) with $a = 1, k = 1$ and get

$$b_{1n} = \frac{4 J_3(\alpha_{1n})}{\alpha_{1n}^2 J_2^2(\alpha_{1n})} = \frac{16}{\alpha_{1n}^3 J_2(\alpha_{1n})},$$

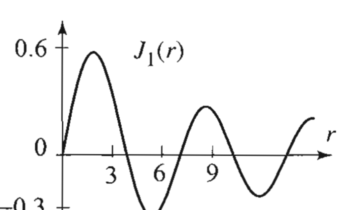

Figure 1 $J_1(r)$ and its first four positive zeros.

where in the last step we have used (6) from Section 4.8 with $p = 2$, and the fact that $J_1(\alpha_{1n}) = 0$. Recall that α_{1n} denotes the nth positive zero of J_1. See Figure 1 for an illustration and Table 1, Section 4.8 for a list of numerical values of the first five α_{1n}. Substituting b_{1n} into (7), we arrive at the solution

$$u(r, \theta, t) = \sin\theta \sum_{n=1}^{\infty} \frac{16}{\alpha_{1n}^3 J_2(\alpha_{1n})} J_1(\alpha_{1n} r) \cos\alpha_{1n} t.$$

With the help of a computer system we found approximate numerical values of the first three coefficients in the series and plotted in Figure 2 the partial sum of the series solution (with n up to 3) at various values of t. ■

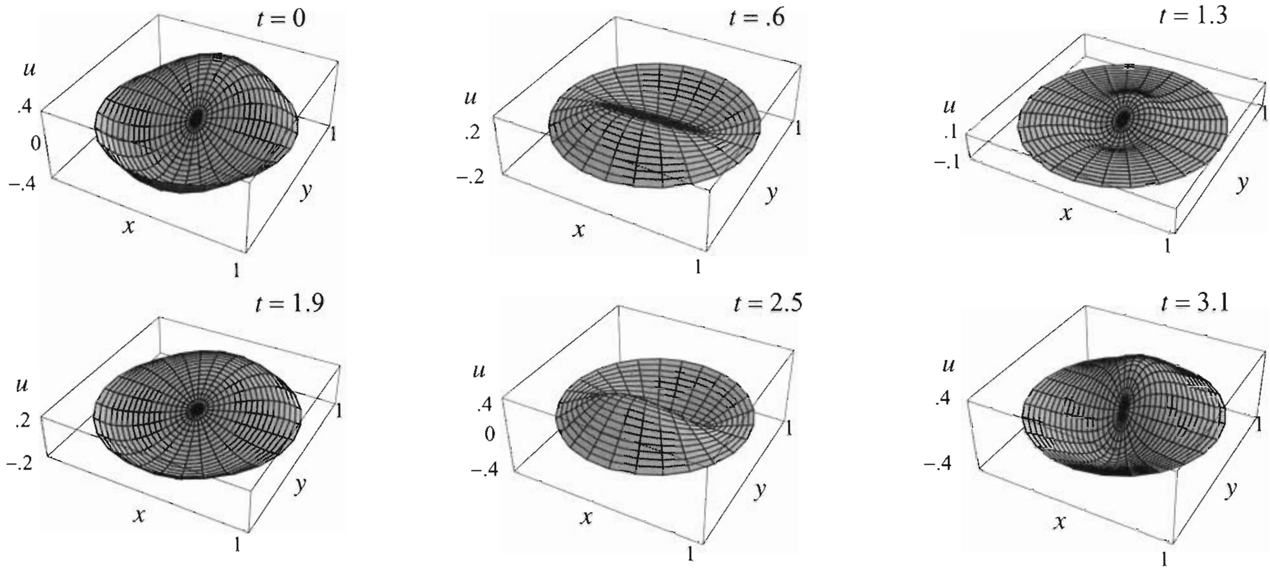

Figure 2 Vibrating circular membrane: a nonradially symmetric case.

To complete the solution of the vibrating membrane, we need to treat the case of a nonzero initial velocity. Save for some minor differences, this case is similar to the one we just treated. The proof is outlined in Exercises 7 and 8. For ease of reference, we state the entire solution for this case in the following box.

THE WAVE EQUATION IN POLAR COORDINATES: GENERAL CASE

The solution of the boundary value problem (1)–(3) is given by

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)(a_{mn}\cos m\theta+b_{mn}\sin m\theta)\cos c\lambda_{mn}t$$

$$(16)\qquad +\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)(a^*_{mn}\cos m\theta+b^*_{mn}\sin m\theta)\sin c\lambda_{mn}t,$$

where $\lambda_{mn}=\frac{\alpha_{mn}}{a}$; α_{mn} is the nth positive zero of J_m; a_{mn}, b_{mn} are given by (12)–(14); and

$$(17)\qquad a^*_{0n}=\frac{1}{\pi c\alpha_{0n}aJ_1^2(\alpha_{0n})}\int_0^a\int_0^{2\pi}g(r,\theta)J_0(\lambda_{0n}r)r\,d\theta\,dr,$$

$$(18)\qquad a^*_{mn}=\frac{2}{\pi c\alpha_{mn}aJ_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi}g(r,\theta)\cos m\theta J_m(\lambda_{mn}r)r\,d\theta\,dr,$$

$$(19)\qquad b^*_{mn}=\frac{2}{\pi c\alpha_{mn}aJ_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi}g(r,\theta)\sin m\theta J_m(\lambda_{mn}r)r\,d\theta\,dr,$$

for $m,\,n=1,\,2,\,\ldots$.

EXAMPLE 3 Nonzero initial displacement and velocity
Determine the solution $u(r,\theta,t)$ of (1)–(3) when

$$a = c = 1,\ f(r,\theta) = (1-r^2)r\sin\theta,\ g(r,\theta) = (1-r^2)r^2\sin 2\theta\,.$$

Solution The solution is given by (16). We only need to compute the second double series since the first one is computed in Example 2. Using the orthogonality of the trigonometric functions and arguing as we did in Example 2, we find that $a^*_{mn} = 0$ for all m and n, and $b^*_{mn} = 0$, except when $m = 2$. To compute b^*_{2n}, we use (19) with $g(r,\theta) = (1-r^2)r^2\sin 2\theta$, $a = c = 1$, and $\lambda_{mn} = \alpha_{mn}$, and get

$$\begin{aligned} b^*_{2n} &= \frac{2}{\pi\alpha_{2n}J_3^2(\alpha_{2n})}\int_0^1\int_0^{2\pi}(1-r^2)r^2\sin^2 2\theta J_2(\alpha_{2n}r)r\,d\theta\,dr \\ &= \frac{2}{\alpha_{2n}J_3^2(\alpha_{2n})}\int_0^1(1-r^2)r^3J_2(\alpha_{2n}r)\,dr, \end{aligned}$$

because $\frac{1}{\pi}\int_0^{2\pi}\sin^2 2\theta\,d\theta = 1$. To compute the last integral we apply (15) with $a = 1$, $k = 2$, and obtain

$$b^*_{2n} = \frac{4J_4(\alpha_{2n})}{\alpha_{2n}^3J_3^2(\alpha_{2n})} = \frac{24}{\alpha_{2n}^4J_3(\alpha_{2n})},$$

where in the last step we have used (6) from Section 4.8 with $p = 3$ and the fact that $J_2(\alpha_{2n}) = 0$. Substituting in the second double series in (16) and using the solution of Example 2, we get the solution

$$\begin{aligned} u(r,\theta,t) &= \sin\theta\sum_{n=1}^{\infty}\frac{16}{\alpha_{1n}^3J_2(\alpha_{1n})}J_1(\alpha_{1n}r)\cos\alpha_{1n}t \\ &\quad + \sin 2\theta\sum_{n=1}^{\infty}\frac{24}{\alpha_{2n}^4J_3(\alpha_{2n})}J_2(\alpha_{2n}r)\sin\alpha_{2n}t\,. \end{aligned}$$

The coefficients in the series can be approximated with the help of a computer, as we did in Example 2. ■

In the exercises, we will use the methods of this section to solve the general heat problem on the disk.

Exercises 4.3

In Exercises 1–8, solve the vibrating membrane problem (1)–(3) for the given data. If possible, with the help of a computer, find numerical values for the first five nonzero coefficients of the series solution and plot the shape of the membrane at various values of t. (Formula (15) is helpful in doing these problems.)

1. $f(r,\theta) = (1-r^2)r^2\sin 2\theta,\ g(r,\theta) = 0,\ a = c = 1.$

2. $f(r,\theta) = (9-r^2)\cos 2\theta,\ g(r,\theta) = 0,\ a = 3,\ c = 1.$

3. $f(r,\theta) = (4-r^2)r\sin\theta,\ g(r,\theta) = 1,\ a = 2,\ c = 1.$

4. $f(r,\theta) = J_3(\alpha_{32}\,r)\sin 3\theta,\ g(r,\theta) = 0,\ a = c = 1.$

5. $f(r,\theta)=0,\ g(r,\theta)=(1-r^2)r^2\sin 2\theta,\ a=c=1.$

6. $f(r,\theta)=1-r^2,\ g(r,\theta)=J_0(r),\ a=c=1.$

7. Project Problem: Circular membrane with zero initial displacement. Follow the steps outlined in this exercise to determine the vibrations of a circular membrane with radius a and fixed boundary, given that the initial displacement of the membrane is 0, and its initial velocity is $g(r,\theta)$. Review the solution of Example 1 for hints.

(a) Write down explicitly the differential equation, the boundary conditions, and the initial conditions.

(b) Assume a product solution of the form $R(r)\Theta(\theta)T(t)$ and show that $T(0)=0$. Conclude that

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)(a^*_{mn}\cos m\theta+b^*_{mn}\sin m\theta)\sin c\lambda_{mn}t\,.$$

(c) Use the given initial velocity and (b) to obtain

$$\begin{aligned} g(r,\theta)=\sum_{n=1}^{\infty}c\lambda_{0n}a^*_{0n}J_0(\lambda_{0n}r)+\sum_{m=1}^{\infty}\Big\{\Big(\sum_{n=1}^{\infty}c\lambda_{mn}a^*_{mn}J_m(\lambda_{mn}r)\Big)\cos m\theta \\ +\Big(\sum_{n=1}^{\infty}c\lambda_{mn}b^*_{mn}J_m(\lambda_{mn}r)\Big)\sin m\theta\Big\}.\end{aligned}$$

(d) Derive (17)–(19) by proceeding from here as we did in the derivation of (12)–(14).

8. General solution of the vibrating circular membrane problem.

(a) Show that the solution of the boundary value problem (1)–(3) can be written as $u(r,\theta,t)=u_1(r,\theta,t)+u_2(r,\theta,t)$, where u_1 and u_2 satisfy (1) and (3) and the following initial conditions:

$$\begin{aligned} u_1(r,\theta,0)&=f(r,\theta), & \tfrac{\partial u_1}{\partial t}(r,\theta,0)&=0;\\ u_2(r,\theta,0)&=0, & \tfrac{\partial u_2}{\partial t}(r,\theta,0)&=g(r,\theta)\,.\end{aligned}$$

(b) Combine the results of Example 1 and Exercise 7 to derive the general solution (16).

9. Project Problem: An integral formula for Bessel functions. Follow the outlined steps to prove that for any $k\ge 0$, and any integer $l\ge 0$, we have

$$\int r^{k+1+2l}J_k(r)\,dr=\sum_{n=0}^{l}(-1)^n\,2^n\frac{l!}{(l-n)!}r^{k+1+2l-n}J_{k+n+1}(r)+C\,.$$

(a) Show that the formula holds for $l=0$ and all $k\ge 0$. [Hint: Use (7), Section 4.8.]

(b) Complete the proof by induction on l. [Hint: Assume the formula is true for $l-1$ and all k. To establish the formula for l, integrate by parts and use the formula with $l-1$ and $k+1$.]