**EXAMPLE 1** Use spherical coordinates to compute the Laplacian of

$$f(x, y, z) = \ln(x^2 + y^2 + z^2), \qquad (x, y, z) \neq (0, 0, 0).$$

**Solution** In spherical coordinates, we have

$$f(r, \theta, \phi) = \ln r^2 = 2\ln r.$$

Since $f$ is independent of $\theta$ and $\phi$, all partial derivatives in these variables are zero. From (8) we get

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} = -\frac{2}{r^2} + \frac{4}{r^2} = \frac{2}{r^2}. \qquad \blacksquare$$

## Exercises 4.1

*In Exercises 1–8, compute the Laplacian in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$. The appropriate dimension is indicated by the number of variables.*

**1.** $u(x, y) = \frac{x}{x^2+y^2}$.

**2.** $u(x, y) = \tan^{-1}(\frac{y}{x})$.

**3.** $u(x, y) = \frac{1}{\sqrt{x^2+y^2}}$.

**4.** $u(x, y, z) = \frac{z}{\sqrt{x^2+y^2}}$.

**5.** $u(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$.

**6.** $u(x, y) = \ln(x^2 + y^2)$.

**7.** $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$.

**8.** $u(x, y) = \tan^{-1}(\frac{y}{x})\frac{y}{x^2+y^2}$.

**9.** (a) Show that if $u(r, \theta, \phi)$ depends only on $r$, then the Laplacian takes the form $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$.
(b) What is the form of the Laplacian if the function $u$ depends only on $r$ and $\theta$?

**10.** Supply all the details to derive (8) from (7).

**11. Project Problem: Harmonic functions.** Recall from Section 3.1 that $u(x, y)$ is called a harmonic function if it satisfies Laplace's equation.
(a) Show that if $u$ and $v$ are harmonic and $\alpha$ and $\beta$ are numbers, then $\alpha u + \beta v$ is harmonic.
(b) Give an example of two harmonic functions $u$ and $v$ such that $u\,v$ is not harmonic.
(c) Show that if $u$ and $u^2$ are both harmonic, then $u$ must be constant. [Hint: Write down what it means for $u$ and $u^2$ to be harmonic in terms of their partial derivatives.]
(d) Show that if $u$, $v$ and $u^2 + v^2$ are harmonic, then $u$ and $v$ must be constant.

## 4.2 Vibrations of a Circular Membrane: Symmetric Case

In this and the next section we study the vibrations of a thin circular membrane with uniform mass density, clamped along its circumference. We place the center of the membrane at the origin, and we denote the radius by $a$. The vibrations of the membrane are governed by the two-dimensional wave equation, which will be expressed in polar coordinates, because these are the coordinates best suited to this problem. Using the polar form of the Laplacian ((3), Section 4.1), the **two dimensional wave equation** becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right). \tag{1}$$

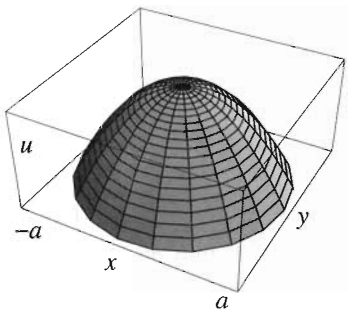



**Figure 1** A radially symmetric shape.

The initial shape of the membrane will be modeled by the function $f(r, \theta)$, and its initial velocity by $g(r, \theta)$.

In this section we confine our study to the case where $f$ and $g$ are **radially symmetric** or **axisymmetric**, that is, they depend only on the radius $r$ and not on $\theta$. It is reasonable on physical grounds that in this case the solution also does not depend on $\theta$ (see Figure 1). Consequently, $\partial u/\partial \theta = 0$, and (1) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{2}$$

where $u = u(r, t)$, $0 < r < a$, and $t > 0$. Since the membrane is clamped at the circumference, we have the **boundary condition**

$$u(a, t) = 0, \qquad t \geq 0. \tag{3}$$

The radially symmetric **initial conditions** are

The initial conditions are radially symmetric, so they depend only on $r$.

$$u(r, 0) = f(r), \quad \frac{\partial u}{\partial t}(r, 0) = g(r), \quad 0 < r < a. \tag{4}$$

We solve the boundary value problem (2)–(4) using the separation of variables method, as we did throughout Chapter 3. The goal is to separate the variables in the partial differential equation (2) and reduce the problem to two ordinary differential equations in $r$ and $t$. As you will see, the equation in $t$ is the same as the one that we obtained after separating variables in the wave equation in rectangular coordinates. Hence the solution in $t$ will consist of sines and cosines. The equation in the spatial variable $r$ is new, and its solution will involve the so-called Bessel functions.

## Separating Variables

We assume that the solution is of the form $u(r, t) = R(r)T(t)$. After differentiating, plugging into (2), and separating variables, we get

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = -\lambda^2.$$

Because we expect periodic solutions in $T$, we have set the sign of the separation constant negative. (For a more rigorous argument based on the fact that the solution in $R$ should be bounded in the interval $[0, a]$, see the solution of the Dirichlet problem in Section 4.5.) Hence

$$rR'' + R' + \lambda^2 rR = 0, \quad R(a) = 0 \qquad \text{(from (3))}, \tag{5}$$

$$T'' + c^2\lambda^2 T = 0\,. \tag{6}$$

## Solving the Separated Equations

Here again, we begin by solving the equation with the boundary conditions to narrow down the possible solutions. Equation (5) is known as the **parametric form of Bessel's equation of order zero** (here $\lambda$ is the parameter). This equation arises so frequently in applications that its solutions have been named. Since the equation is second order and homogeneous, we need only two linearly independent solutions to be able to write its general solution. By convention, these two linearly independent solutions are called **Bessel functions of order 0 of the first** and **second kind**, and are denoted $J_0(\lambda r)$ and $Y_0(\lambda r)$, respectively. Hence the general solution to the parametric form of Bessel's equation in (5) is

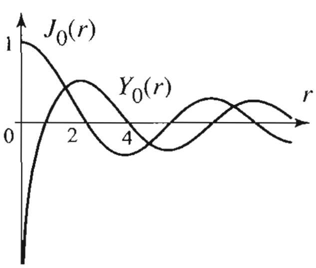



**Figure 2** The Bessel functions of order 0.

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)\,, \tag{7}$$

where $r > 0$ (Theorem 3, Section 4.8). The functions $J_0$ and $Y_0$ are treated in great detail in Sections 4.7-4.9; here we recall facts only as needed. Figure 2 shows the graphs of $J_0$ and $Y_0$ .

Since on physical grounds the solutions to the wave equation are expected to be bounded, it follows that the spatial part of the solution, $R(r)$, has to be bounded near $r = 0$. This is effectively a second boundary condition on $R$. Now the fact that $Y_0$ is unbounded near 0 forces us to choose $c_2 = 0$ in (7). To avoid trivial solutions, we will take $c_1 = 1$ and get

$$R(r) = J_0(\lambda r)\,. \tag{8}$$

The condition $R(a) = 0$ (see (5)) implies that

$$J_0(\lambda a) = 0,$$

and so $\lambda a$ must be a root of the Bessel function $J_0$. As Figure 2 suggests, $J_0$ has infinitely many positive zeros, which we denote by

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \cdots .$$

(For a proof of this fact, see Section 4.9, or Exercise 35, Section 4.8.) Thus

$$\lambda = \lambda_n = \frac{\alpha_n}{a}, \quad n = 1, 2, \ldots,$$

and the corresponding solutions of (5) are

$$\boxed{R_n(r) = J_0\left(\frac{\alpha_n}{a}r\right), \quad n = 1, 2, \ldots,}$$

where $\alpha_n$ is the $n$th positive zero of $J_0$. These solutions are analogous to the solutions $\sin\frac{n\pi}{L}x$ that we have encountered several times previously, in particular, while solving the one dimensional wave equation. The only difference is that the function sine and its zeros $n\pi$ are now replaced by the function $J_0$ and its zeros $\alpha_n$. Returning to (6) with $\lambda = \lambda_n$ , we find

$$\boxed{T(t) = T_n(t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t\,.}$$

We thus obtain the product solutions of (2) and (3)

$$u_n(r,t) = (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) J_0(\lambda_n r) \quad n = 1, 2, \ldots .$$

## Bessel Series Solution of the Entire Problem

To satisfy the initial conditions, motivated by the superposition principle, we let

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) J_0(\lambda_n r)\,.$$

We determine the unknown coefficients by evaluating the series at $t = 0$ and using the initial conditions. We get from the first condition in (4)

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r), \quad 0 < r < a\,.$$

This series representation of $f(r)$ is akin to a Fourier sine series, except that the sine functions are now replaced by Bessel functions. There are analogous expansion theorems that apply in such cases; the series expansions that arise are known as **Bessel**, or **Fourier-Bessel, expansions** (see Theorem 2, Section 4.8). For the case at hand, we make use of Theorem 2, Section 4.8, with $p = 0$. The **Bessel coefficients** $A_n$ are given by

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r) r\, dr,$$

where $J_1$ is the Bessel function of order 1. Now, differentiating the series for $u$ term by term with respect to $t$, and then setting $t = 0$, we get from the second initial condition

$$u_t(r,0) = g(r) = \sum_{n=1}^{\infty} c\lambda_n B_n J_0(\lambda_n r)\,.$$

Thus $c\lambda_n B_n = c\frac{\alpha_n}{a} B_n$ is the $n$th Bessel coefficient of $g$, and so

$$B_n = \frac{2}{c\,\alpha_n\, a J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r)\, r\, dr\,.$$

This completely determines the solution.

**THEOREM 1 WAVE EQUATION IN POLAR COORDINATES**

The solution of the radially symmetric two-dimensional wave equation (2) with boundary and initial conditions (3) and (4) is

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) J_0(\lambda_n r), \tag{9}$$

where

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r) r\, dr,$$

(10)

$$B_n = \frac{2}{c\,\alpha_n a\, J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r)\, r\, dr;$$

$$\lambda_n = \frac{\alpha_n}{a}, \quad \text{and} \quad \alpha_n = n\text{th positive zero of } J_0\,.$$

There is a clear analogy between the solution (9) and the solution of the one-dimensional wave equation (8), Section 3.3. The only difference is that spatial variations are now determined by Bessel functions rather than the simpler sine functions.

When applying (10) in concrete situations, we are required to evaluate integrals involving Bessel functions that are quite complicated. In many interesting cases these integrals can be evaluated with the help of integral formulas developed in the exercises and in Section 4.8. As an illustration, consider the integral

$$\int_0^a x^{p+1} J_p\left(\frac{\alpha}{a}x\right) dx, \quad p \geq 0,\ \alpha > 0.$$

Let $u = \frac{\alpha}{a}x$, $du = \frac{\alpha}{a}dx$, then

$$\begin{aligned}\int x^{p+1} J_p\left(\frac{\alpha}{a}x\right) dx &= \frac{a^{p+2}}{\alpha^{p+2}} \int u^{p+1} J_p(u)\, du \\ &= \frac{a^{p+2}}{\alpha^{p+2}} u^{p+1} J_{p+1}(u) + C,\end{aligned}$$

From (7), Section 4.8,

$$\int x^{p+1} J_p(x)\, dx = x^{p+1} J_{p+1}(x) + C\,.$$

where the last equality follows from (7), Section 4.8. Substituting back $u = \frac{\alpha x}{a}$, simplifying, and then evaluating at $x = 0$ and $x = a$, we obtain the very useful identity

$$\int_0^a x^{p+1} J_p\left(\frac{\alpha}{a}x\right) dx = \frac{a^{p+2}}{\alpha} J_{p+1}(\alpha)\,. \tag{11}$$

**EXAMPLE 1 A circular membrane with constant initial velocity**

An explosion near the surface of a flexible circular membrane with clamped edges imparts a uniform initial velocity equal to $-100$ m/sec. Assume the initial shape of the membrane to be flat, take $a = 1$ and $c = 100$, and determine the subsequent vibrations of the membrane.

**Solution** The solution is given by (9), where $A_n = 0$ for all $n$, since $f(r) = 0$. From (10) we have

$$\begin{aligned} B_n &= \frac{-2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_n r)\, r\, dr \\ &= \frac{-2}{\alpha_n^2 J_1(\alpha_n)} \qquad \text{(by (11) with } p = 0\text{)}. \end{aligned}$$

Thus, from (9), we obtain the solution

$$u(r,t) = \sum_{n=1}^{\infty} \frac{-2}{\alpha_n^2 J_1(\alpha_n)} \sin(100\, \alpha_n t) J_0(\alpha_n r)\,. \qquad \blacksquare$$

To get numerical values from our answer in Example 1, it is clearly necessary to know the values of the zeros of the Bessel function $J_0$. Since these values are useful in solving many problems, they have been computed and tabulated to a high degree of accuracy. With the help of a computer system, we approximated the first five positive roots of the equation $J_0(x) = 0$. These and other relevant numerical data are given in Table 1.

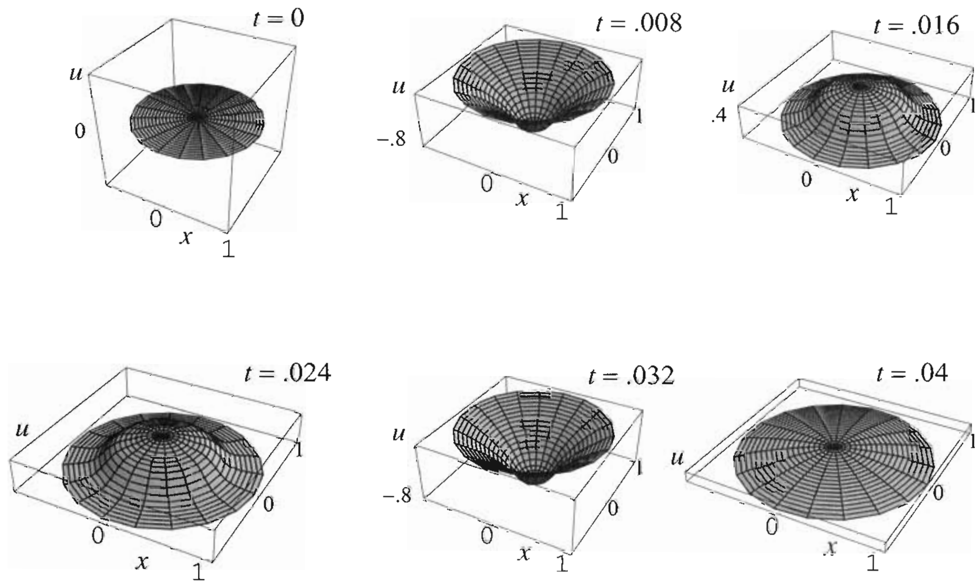
| $j$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\alpha_j$ | 2.4048 | 5.5201 | 8.6537 | 11.7915 | 14.9309 |
| $J_1(\alpha_j)$ | .5191 | $-.3403$ | .2714 | $-.2325$ | .2065 |
| $\frac{-2}{\alpha_j^2 J_1(\alpha_j)}$ | $-0.6662$ | 0.1929 | $-.0984$ | 0.0619 | $-0.0434$ |

**Table 1** Numerical data for Example 1.

With the help of this table, we find the first three terms of the solution in Example 1:

$$\begin{aligned} u(r,t) \approx\ & -0.6662\, J_0(2.40\, r) \sin(240\, t) \\ & +0.1929\, J_0(5.52\, r) \sin(552\, t) - .0984\, J_0(8.65\, r) \sin(865\, t)\,. \end{aligned}$$

We used these terms to plot the shape of the membrane at various values of $t > 0$ in Figure 3.




As expected, soon after the explosion, the elastic membrane starts to vibrate downward.

**Figure 3** Vibrating circular membrane with radial symmetry in Example 1.

The next example treats the case of a vibrating membrane with nonzero initial displacement and zero initial velocity.

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**EXAMPLE 2 A circular membrane with radially symmetric initial shape**
Solve the boundary value problem (2)–(4), given that

$$f(r) = 1 - r^2, \quad g(r) = 0, \quad a = c = 1.$$

**Solution** Note that the problem is radially symmetric because of the boundary and initial conditions. The solution is given by (9), where $B_n = 0$ for all $n$ since $g(r) = 0$, and $A_n$ is the Bessel coefficient of the function $1 - r^2$, given by (10). We have

$$\begin{aligned} A_n &= \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1 - r^2) J_0(\alpha_n r)\, r\, dr \\ &= \frac{2}{\alpha_n^4\, J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s) s\, ds \qquad (s = \alpha_n r). \end{aligned}$$

Integrating by parts, with $u = \alpha_n^2 - s^2$, $dv = J_0(s)s\, ds$, and hence $du = -2s\, ds$, $v = J_1(s)s$ (by (7), Section 4.8, with $p = 0$), we find

$$\begin{aligned} A_n &= \frac{2}{\alpha_n^4\, J_1^2(\alpha_n)} \left[ (\alpha_n^2 - s^2) J_1(s) s \Big|_0^{\alpha_n} + 2 \int_0^{\alpha_n} J_1(s) s^2\, ds \right] \\ &= \frac{4}{\alpha_n^4\, J_1^2(\alpha_n)} \int_0^{\alpha_n} J_1(s) s^2\, ds\, . \end{aligned}$$

To evaluate the integral, we appeal to (11) and arrive at

$$A_n = \frac{4}{\alpha_n^2\, J_1^2(\alpha_n)} J_2(\alpha_n)\, .$$

At this point, we could appeal to (9) to write the explicit form of the solution. However, before doing so, we mention one more worthy simplification based on yet another property of Bessel functions. Appealing to (6), Section 4.8, with $p = 1$, and recalling that $\alpha_n$ is a zero of $J_0$, we find

From (6), Section 4.8, $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$.

$$J_2(\alpha_n) = \frac{2}{\alpha_n} J_1(\alpha_n),$$

and hence

$$A_n = \frac{8}{\alpha_n^3 J_1(\alpha_n)}.$$

Thus, from (9), we obtain the solution

$$u(r,t) = \sum_{n=1}^{\infty} \frac{8}{\alpha_n^3 J_1(\alpha_n)} \cos(\alpha_n t) J_0(\alpha_n r). \qquad \blacksquare$$

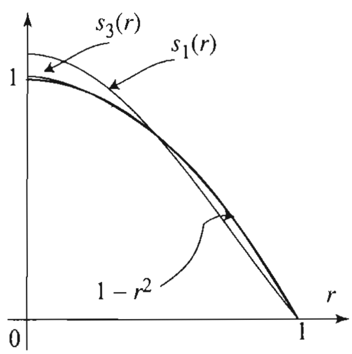

**Figure 4** Partial sums of Bessel series.

Setting $t = 0$ in the solution of Example 2, we should get the initial displacement, that is, we should get

$$1 - r^2 = \sum_{n=1}^{\infty} \frac{8}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n r), \quad 0 < r < 1.$$

This is the Bessel series of the function $1 - r^2$ that we have computed in passing as we worked out the solution to Example 2. Figure 4 shows some partial sums of this series converging to $1 - r^2$, $0 < r < 1$.

We end this section with a remark concerning the physical interpretation of the solution of Example 2. In our derivation of the wave equation, we always assumed small displacements, but you may not be willing to call a unit displacement at the center of a drum of unit radius small. To give our problem a meaningful interpretation, we could rescale the initial data. Because of the linearity of the boundary value problem this leads only to the same rescaling of the solution.

## Exercises 4.2

*In Exercises 1–8, solve the vibrating membrane problem (2)–(4) for the given data. If possible, with the help of a computer, find numerical values for the first five nonzero coefficients of the series solution and plot the shape of the membrane at various values of $t$. (Formula (11) is useful in all these exercises.)*

**1.** $a = 2,\ c = 1,\ f(r) = 0,\ g(r) = 1.$

**2.** $a = 1,\ c = 10,\ f(r) = 1 - r^2,\ g(r) = 1.$

**3.** $a = 1,\ c = 1,\ f(r) = 0,$

$$g(r) = \begin{cases} 1 & \text{if } 0 < r < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < r < 1. \end{cases}$$

[Hint: Follow Example 1.]