Homework 8 solutions

7.2.36

(a) $f(x) = e^{-\frac{1}{2}x^2+2} = e^2 e^{-\frac{1}{2}x^2}$

$\hat{f}(\omega) = e^2 \widehat{e^{-\frac{1}{2}x^2}}(\omega) = e^2 e^{-\frac{1}{2}\omega^2}$, by using Theorem 7.2.5

(b) $f(x) = e^{-2x^2+2x} = e^{-2(x-\frac{1}{2})^2+\frac{1}{2}}$ (completing squares)

$\hat{f}(\omega) = e^{\frac{1}{2}} \mathcal{F}\left[e^{-2(x-\frac{1}{2})^2}\right](\omega)$

$= e^{\frac{1}{2}} e^{-i\omega/2} \mathcal{F}\left[e^{-2x^2}\right](\omega)$ (shifting property)

$\boxed{= e^{\frac{1}{2}} e^{-i\omega/2} \frac{1}{2} e^{-\omega^2/8}}$ (theorem 7.2.5 with $a=4$)

7.2.43:

$f(x) = (1-x^2)e^{-x^2}$

$\mathcal{F}(f)(\omega) = \mathcal{F}(e^{-x^2})(\omega) - \mathcal{F}(x^2 e^{-x^2})(\omega)$

Using theorem 7.2.5: $\mathcal{F}(e^{-x^2})(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$

Moreover theorem 3 ii gives:

$$\mathcal{F}(x^2 e^{-x^2})(\omega) = i^2 \frac{d^2}{d\omega^2}\left[\mathcal{F}(e^{-x^2})(\omega)\right]$$

$$= -\frac{1}{\sqrt{2}} \frac{d^2}{d\omega^2}\left(e^{-\omega^2/4}\right)$$

$$= -\frac{1}{\sqrt{2}}\left(-\frac{1}{2} + \frac{1}{4}\omega^2\right)$$

$$\Rightarrow \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}\left(1 - \frac{1}{2} + \frac{1}{4}\omega^2\right)$$

$$\boxed{= \frac{1}{4\sqrt{2}} e^{-\omega^2/4}\left(2 + \omega^2\right)}$$

7.2.50 $\hat{h}(\omega) = e^{-\omega^2} \frac{\sin\omega}{\omega}$

$$e^{-\omega^2} = \mathcal{F}\left(\frac{1}{\sqrt{2}} e^{-x^2/4}\right)(\omega)$$

$$\frac{\sin\omega}{\omega} = \sqrt{\frac{\pi}{2}}\, \mathcal{F}(f(x))(\omega) \quad \text{where } \boxed{f(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1 \end{cases}}$$

$$\Rightarrow h(x) = f(x) * g(x)$$

$$\text{where } \boxed{g(x) = \frac{1}{\sqrt{2}} e^{-x^2/4}}$$

7.3.9 Solve $\begin{cases} t^2 u_x - u_t = 0 \\ u(x,0) = \underbrace{3\cos x}_{f(x)} \end{cases}$

technicality: Fourier transform of cos x not defined as function is not integrable over $\mathbb{R}$

depends on version of book, approach is similar.

$\mathcal{F}$ w.r.t x

$$\begin{cases} t^2 i\omega \hat{u}(\omega,t) - \frac{d}{dt}\hat{u}(\omega,t) = 0 \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

solving: $\hat{u}(\omega,t) = A(\omega)\, e^{-\frac{i\omega t^3}{3}}$

$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-\frac{i\omega t^3}{3}}$ (using I.C.)

$$\Rightarrow \boxed{u(x,t)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-\frac{i\omega t^3}{3}} e^{ix\omega}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega(x + t^3/3)}\, d\omega$$

$$= \boxed{f\left(x+\frac{t^3}{3}\right) = \cos\left(x+\frac{t^3}{3}\right)}$$

7.3.18

$$\begin{cases} u_t = t\, u_{xxxx} \\ u(x,0) = f(x) \end{cases}$$

$$\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = t\,(i\omega)^4\,\hat{u}(\omega,t) = t\,\omega^4\,\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\hat{u}(\omega,t) = A(\omega)\, e^{\omega^4 t^2/2}$$

$$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{\omega^4 t^2/4} \quad \text{(using } t=0\text{)}$$

7.4.3

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = 70e^{-x^2/2} \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = -\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = 70e^{-\omega^2/2} \end{cases}$$

$$\Rightarrow \hat{u}(\omega,t) = A(\omega)e^{-\omega^2 t}$$

$$\hat{u}(\omega,0) = A(\omega) = 70e^{-\omega^2/2}$$

$$\Rightarrow \hat{u}(\omega,t) = 70e^{-\omega^2/2}e^{-\omega^2 t} = 70e^{-\omega^2(t+\frac{1}{2})}$$

$$u(x,t) = \mathcal{F}^{-1}\left(70e^{-\omega^2(t+\frac{1}{2})}\right)(\omega) \qquad \begin{cases} \frac{1}{2a} = t+\frac{1}{2} \\ a = \frac{1}{2t+1} \end{cases}$$

$$= \frac{70}{\sqrt{2t+1}}\,\mathcal{F}^{-1}\left(\underbrace{\sqrt{2t+1}}_{\frac{1}{\sqrt{a}}}\, e^{-\omega^2/2a}\right)$$

$$\boxed{= \frac{70}{\sqrt{2t+1}}\exp\left[-\frac{x^2}{2(2t+1)}\right]}$$