# MATH3150 HW4-like exercises

**3.4.3** Use d'Alembert's method to solve the 1D WEQ with

$f(x) = \sin \pi x + 3 \sin 2\pi x$

$g(x) = \sin \pi x$

$c = 1$

$L = 1$

For these functions $f^*(x) = f(x)$ = odd extension of $f$

$g^*(x) = g(x)$ = odd ext. of $g$

D'Alembert's method gives:

$$u(x,t) = \frac{1}{2}\left[f^*(x-t) + f^*(x+ct)\right] + \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$

where $G(x) = \int_0^x g(x)dx = \int_0^x \sin \pi x \, dx$

$$= -\frac{1}{\pi}\cos \pi t \Big|_{t=0}^{x} = \frac{1}{\pi}(1 - \cos \pi x)$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\sin(\pi(x+t)) + 3\sin(2\pi(x+t)) + \sin(\pi(x-t)) + 3\sin(2\pi(x+t\right.$$
$$+ \frac{1}{2\pi}\left[1 - \cos\pi(x+t) - (1 - \cos\pi(x-t))\right]$$

What is the period of the motion?

$f(x)$ is clearly 2-periodic

$G(x)$ as well,

thus:

$$\boxed{u(x,t+2)} = \frac{1}{2}\left[f^*(x-t-2) + f^*(x+t+2)\right]$$
$$+ \frac{1}{2}\left[G(x+t+2) - G(x-t-2)\right]$$
$$= \frac{1}{2}\left[f^*(x-t) + f^*(x+t)\right] + \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$
$$\boxed{= u(x,t).}$$

$\boxed{3.5.6}$ Solve 1D Heat Eq:

$$(A)\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = e^{-x} = f(x) \end{cases}$$

We have $u(x,t) = \sum_{n=1}^{\infty} b_n \exp[-(n\pi)^2 t] \sin(n\pi x)$

where $b_n$ are coeff in Sine series of $e^{-x}$:

$$b_n = 2\int_0^1 e^{-x} \sin(n\pi x)\, dx$$

using last formula on backpage of book.

$$= 2\frac{e^{-x}}{1+(n\pi)^2}\left(-\sin n\pi x - n\pi \cos n\pi x\right)\Big|_0^1.$$

$$= \frac{2e^{-1}}{1+(n\pi)^2}\left(-n\pi(-1)^n\right) - \frac{2}{1+(n\pi)^2}(-n\pi)$$

$$\boxed{= \frac{2n\pi}{1+(n\pi)^2}\left(1 - e^{-1}(-1)^n\right)} \quad (*)$$

Solve 1D Heat eq using solution to (A).

$$(B)\begin{cases} u_t = u_{xx} \\ u(0,t) = 0 \\ u(1,t) = 1 \\ u(x,0) = e^{-x} + x = g(x) \end{cases}$$

First we find steady state solution $s(x)$:

$s(x) = ax + b$ (since $s_{xx}(x) = 0$)

$s(0) = 0 = b$

$s(1) = a = 1$ $\Rightarrow \boxed{s(x) = x}$

Then we look at $v(x,t) = u(x,t) - s(x)$

$v$ solves the DE:

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = g(x) - s(x) = e^{-x} = f(x) \end{cases}$$

this is problem (A).

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-(n\pi)^2 t\right] \sin(n\pi x)$$

where $b_n$ is given as in (*)

We get back to $u(x,t)$ by shifting it back:

$$u(x,t) = v(x,t) + s(x)$$

$$= x + \sum_{n=1}^{\infty} b_n \exp\left[-(n\pi)^2 t\right] \sin(n\pi x).$$