HW4 Solutions

3.4.1

$f(x) = \sin \pi x,\ g(x) = 0,\ c = \frac{1}{\pi}$ for $x \in [0,1]$

f is odd an 2-periodic → no need to do odd-extension.

$$u(x,t) = \frac{1}{2}\left[\sin\left(\pi\left(x + \frac{t}{\pi}\right)\right) + \sin\left(\pi\left(x - \frac{t}{\pi}\right)\right)\right]$$

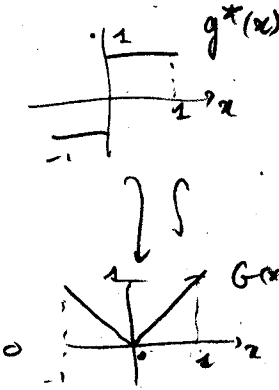
$$= \frac{1}{2}\left[\sin(\pi x + t) + \sin(\pi x - t)\right]$$

3.4.4

$f(x) = 0,\ g(x) = 1,\ c = 1$

For $x \in [-1, 1]$, we have

$$g^*(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ -1 & \text{if } -1 \le x < 0 \end{cases}$$



$$G(x) = \int_{-1}^{x} g^*(s)\,ds = \begin{cases} x & \text{if } 0 \le x \le 1 \\ -x & \text{if } -1 < x < 0 \end{cases}$$

$\Rightarrow u(x,t) = \frac{1}{2}\left(G(x+t) - G(x-t)\right)$ where G has been 2-periodized.

3.4.9

$$u(x, t+2\pi) = \frac{1}{2}\left[\sin(\pi x + t + 2\pi) + \sin(\pi x - t - 2\pi)\right]$$

$$= \frac{1}{2}\left[\sin(\pi x + t) + \sin(\pi x - t)\right]$$

$= u(x,t)$ ⇒ u is 2π-periodic

⇒ The string will pass through it's initial data every 2π (units of time).

$\boxed{3.5.4}$ Solve $\begin{cases} u_t = u_{xx} \\ u(0,t) = u(\pi, t) = 0 \\ u(x,0) = f(x) = \begin{cases} 100 & \text{if } 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases} \end{cases}$

We have $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi/2} 100 \sin nx \, dx = -\frac{200}{\pi} \frac{\cos nx}{n} \Big|_{x=0}^{\pi/2}$$

$$= \boxed{\frac{200}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right)}$$

$\boxed{3.5.14}$ Solve $\begin{cases} u_t = u_{xx} \\ u(0,t) = 0, \ u(\pi, t) = 100 \\ u(x,0) = f(x) \text{ as above} \end{cases}$

the steady state temperature distribution is $\boxed{s(x) = \frac{100}{\pi} x}$

(linear s.t. $s(0) = 0$, $s(\pi) = 100$)

then $v(x,t) = u(x,t) - s(x)$ solves:

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = v(\pi, t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases}$$

$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$, where:

$$b_n = \frac{2}{\pi} \int_0^{\pi} (f(x) - s(x)) \sin nx \, dx = \underbrace{\frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx}_{\text{from } 3.5.4} - \frac{2}{\pi} \int_0^{\pi} \frac{100 x}{\pi} \sin nx$$

$$= \frac{200}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right) - \frac{200}{\pi^2} \left[\cancel{\frac{1}{n^2} \sin nx} - \frac{x}{n} \cos nx \right]_0^{\pi}$$

$$= \frac{200}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right) + \frac{200}{\pi^2} \left(\frac{(-1)^n \pi}{n} \right)$$

$$= \frac{200}{n\pi} \left(1 + (-1)^n - \cos \frac{n\pi}{2} \right) = \boxed{\frac{200}{n\pi} \left(2 - (-1)^n \right)}$$

n	$(-1)^n - \cos \frac{n\pi}{2}$	$1 + ($
0	0	1
1	-1	0
2	2	3
3	-1	0

Thus $v(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}\left(2-(-1)^n\right) e^{-n^2 t} \sin nx$

$$\boxed{\begin{aligned} u(x,t) &= s(x) + v(x,t) \\ &= \frac{100x}{\pi} + \sum_{n=1}^{\infty} \frac{200}{n\pi}\left(2-(-1)^n\right) e^{-n^2 t} \sin nx \end{aligned}}$$

$\boxed{3.4.9}$ (a) $s(x) = 100x$ since $s(0) = 0,\ s(1) = 100$

(b) $s(x) = 100$ since $s(0) = 100$ & $s(1) = 100$.

$\boxed{3.6.2}$ Solve $\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = \cos \pi x \end{cases}$

The initial temp distrib is already given as its Cosine series

thus: $a_1 = 1$ and $a_n = 0$ for $n \neq 1$.

$\Rightarrow \boxed{u(x,t) = \cos \pi x \exp\left[-(\pi)^2 t\right]}$