MATH 3150 HW2

2.1.7 Sums of periodic functions

f_i, $i = 1..n$ are T- per. functions

then: $$\sum_{i=1}^{n} a_i f_i(x+T) = \sum_{i=1}^{n} a_i f_i(x)$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} a_i f_i(x) \text{ is T- periodic.}}$$

The series converges means:

$$\sum_{n=1}^{\infty} a_i f_i(x) = \lim_{N\to\infty} \sum_{i=1}^{N} a_i f_i(x)$$

thus: $$\sum_{n=1}^{\infty} a_i f_i(x+T) = \lim_{N\to\infty} \sum_{i=1}^{N} a_i f_i(x+T)$$
$$= \lim_{N\to\infty} \sum_{i=1}^{N} a_i f_i(x) = \sum_{i=1}^{\infty} a_i f_i(x)$$

$$\Rightarrow \boxed{\sum_{i=1}^{\infty} a_i f_i(x) \text{ is T- periodic}}$$

2.1.9 (a) $f(x+T)\,g(x+T) = f(x)\,g(x) \Rightarrow$ product of T- per functions is T- per

$$\frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)} \Rightarrow$$ quotient of T- per functions is T- per.

(b) Let f be T- periodic.

Then: $f\left(\frac{x+aT}{a}\right) = f\left(\frac{x}{a}+T\right) = f\left(\frac{x}{a}\right)$

thus $f\left(\frac{x}{a}\right)$ is aT- periodic

(c) Let f be a T- periodic function and g some other function.

then: $g(f(x+T)) = g(f(x)) \Rightarrow$ $g(f(x))$ has period T.

$\boxed{2.2.7}$

(a) $f(x) = |\sin x| \quad -\pi \le x \le \pi$

Fourier coeff:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\, dx = \frac{2}{2\pi}\int_0^{\pi} \sin x \, dx = -2 \cos x \Big|_0^{\pi} = \frac{4}{2\pi} = \boxed{\frac{2}{\pi}}$$

$\boxed{a_1 = 0}$ (orthogonality)

for $n \geq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\sin x| \cos nx}_{\text{even fun.}}\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

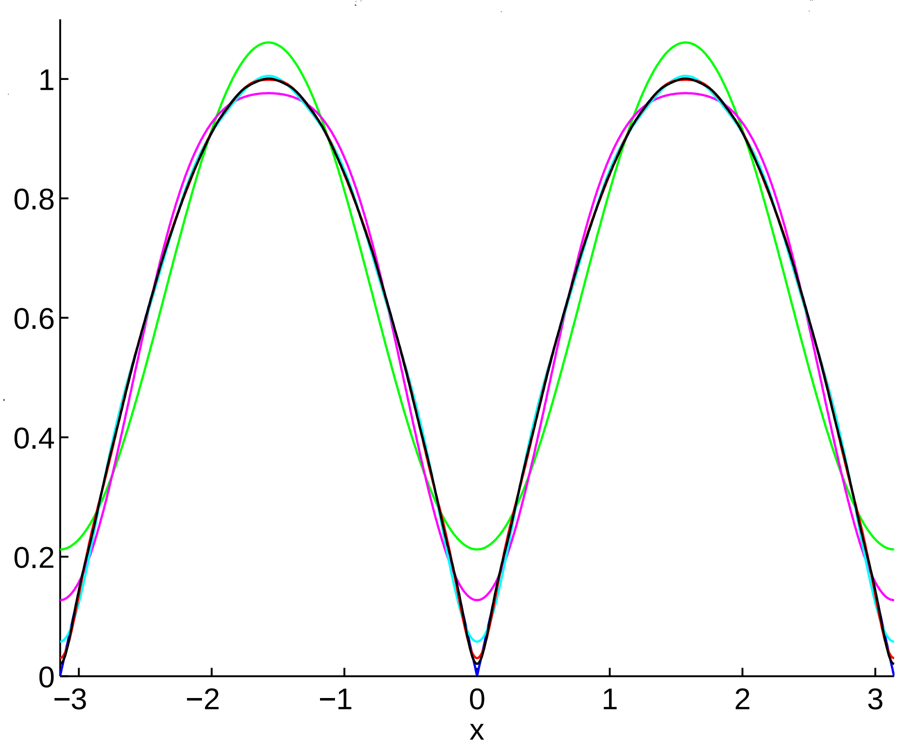
$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(x+nx) + \sin(x-nx)\right] dx = -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(1-n)x}{(1-n)}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{2(-1)^{n+1}}{1-n^2} - \frac{2}{1-n^2}\right] = \begin{cases} \dfrac{4}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\sin x| \sin nx}_{\text{odd}}\, dx = 0.$$

$$\Rightarrow \boxed{f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{1-(2k)^2} \cos 2kx}$$

(b)



2.2.9

(a) $f(x) = x^2$ if $-\pi < x < \pi$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\frac{2\pi^3}{3} = \frac{\pi^2}{3}}$$

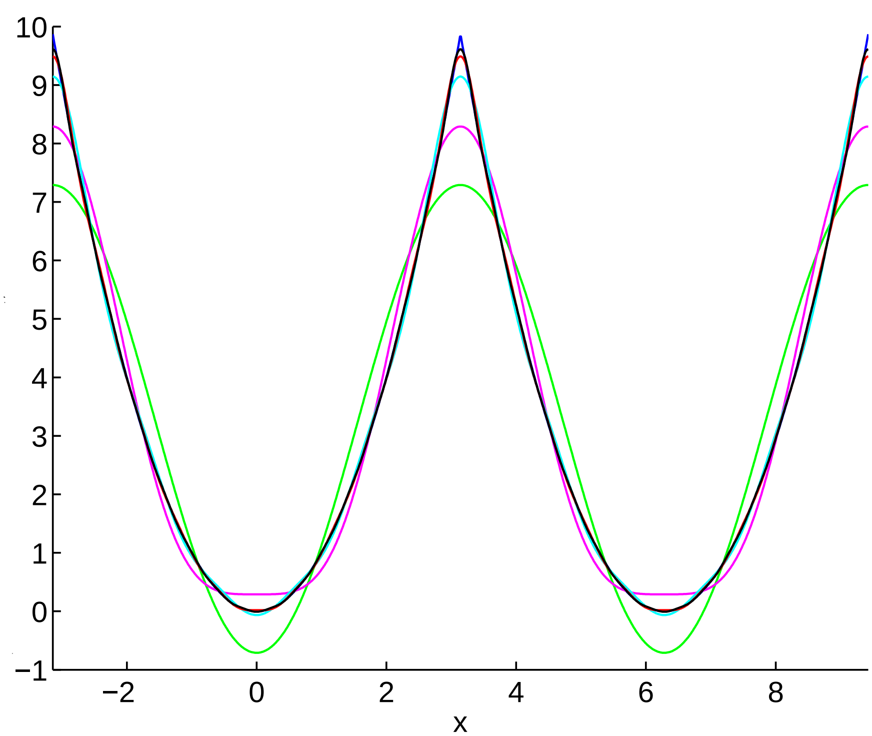
$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2 \sin nx}_{\text{odd function}}\,dx = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \underset{\substack{\uparrow \\ \text{using formula} \\ \text{back of book}}}{=} \frac{1}{\pi}\left[\frac{2x\cos nx}{n^2} + \underbrace{\frac{n^2x^2-2}{n^3}\sin nx}_{=0}\right]_{-\pi}^{\pi}$$

$$\boxed{= \frac{4(-1)^n}{n^2}}$$

$$\Rightarrow \boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx}$$

(b)



2.2.10

(a) $f(x) = 1 - \sin x + 3\cos 2x$

using orthogonality relations:

$$a_0 = \frac{(f,1)}{(1,1)} = \frac{(1,1)}{(1,1)} = 1$$

$$a_n = \frac{(f,\cos nx)}{(\cos nx,\cos nx)} = 3\,\frac{(\cos 2x,\cos 2x)}{(\cos 2x,\cos 2x)} \quad \text{for } n \geq 1$$

$$b_n = \frac{(f,\sin nx)}{(\sin nx,\sin nx)} = \frac{-(\sin x,\sin x)}{(\sin x,\sin x)} = -1 \quad \text{for } n \geq 1$$

Thus $f(x) = 1 - \sin x + 3\cos 2x$

2.2.11

(a) $f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (trig formulas)

using similar reasoning as above: $a_0 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$

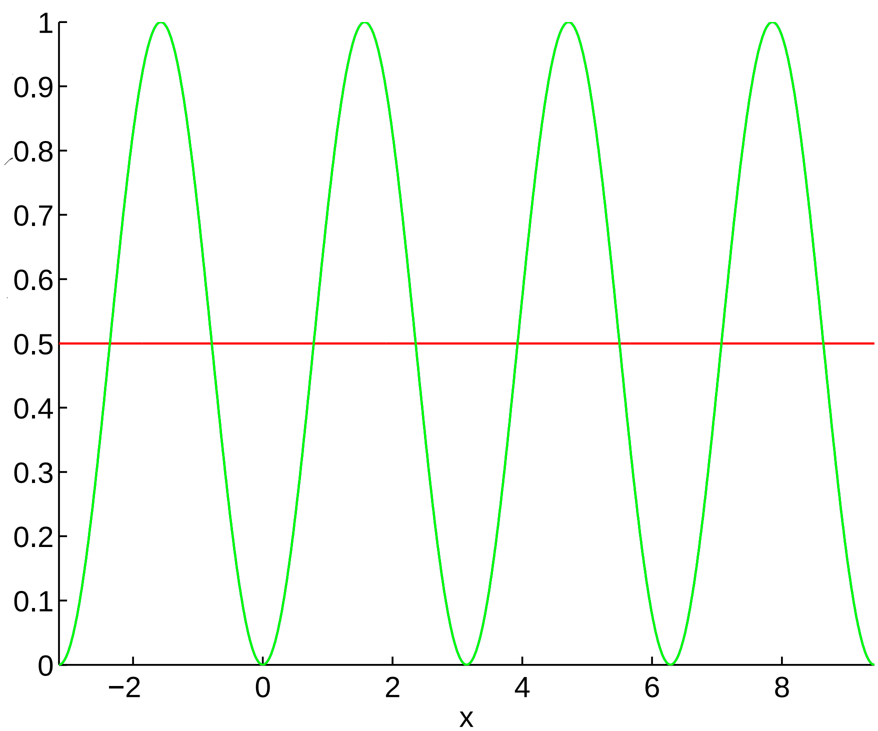
$a_n = 0$ for $n \in \mathbb{N} \setminus \{0, 2\}$

$b_n = 0$ for $n \geq 1$.

$f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\Rightarrow a_0 = \frac{1}{2}$, $a_2 = \frac{1}{2}$ and all other Fourier coeff are zero.

(b)



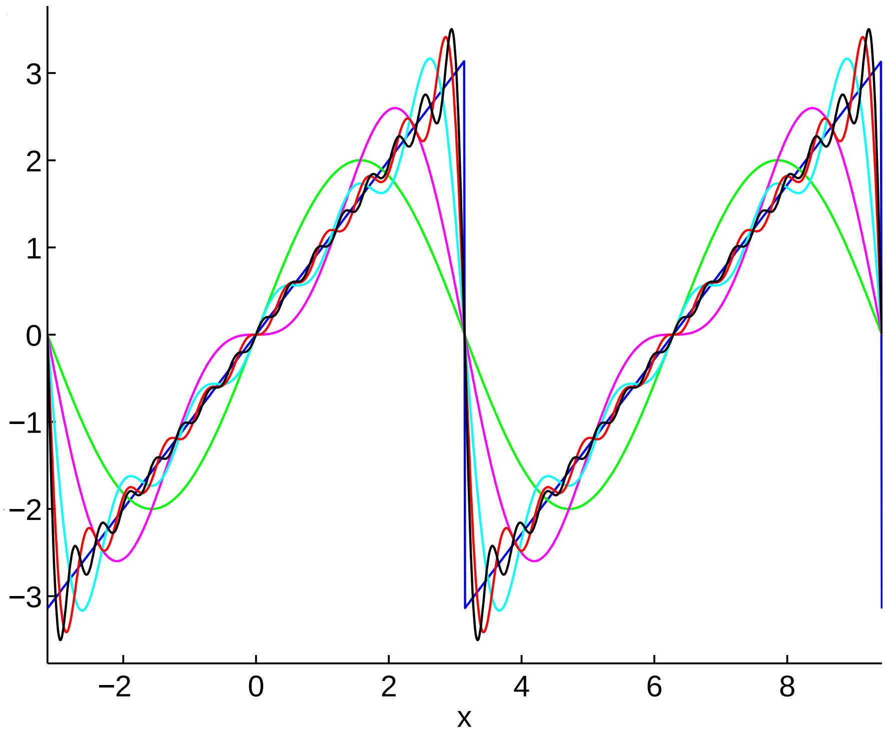
2.2.13

(a) $f(x) = x$ if $-\pi < x < \pi$

we have $f(x) = 2g(\pi - x)$ where $g(x)$ is defined in example 1.

thus:
$$f(x) = 2g(\pi - x) = 2\sum_{n=1}^{\infty} \frac{\sin n(\pi - x)}{n}$$
$$= -2\sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n}.$$

(b)



```
% MATH 3150 Fall 2008
%  problem 2.2.7

thickLines(3); % remove if it is not in your
    system
figure(1); clf;
x = linspace(-pi,pi,1000);

% plot true function for refernce
hold on;
plot(x,abs(sin(x))); % function is 2*pi periodic
    already
axis([-pi,pi,0,1.1]);

% loop over number of terms
Ns = [1,2,5,10,15];
cols={'g','m','c','r','k'};
for iN = 1:length(Ns),
 N=Ns(iN);

 % compute partial Fourier series
 s=2/pi*ones(size(x));
 for k=1:N,
  an = 4/pi/(1-(2*k)^2);
  s=s+an*cos(2*k*x);
 end;

 % comparative plot
 plot(x,s,cols{iN});
end;%N
hold off;
xlabel('x');
filename='p2_2_7.eps';
print('-depsc2',filename);
system(['epstopdf ',filename]);
```

```
% MATH 3150 Fall 2008
% Problem 2.2.9

thickLines(3); % remove if you do not have it in
    your system
figure(1); clf;
x = linspace(-pi,3*pi,1000);

% plot true function for refernce
hold on;
% trick to make 2*pi periodic function from fn
    def on [0,2*pi]
plot(x,(mod(x+pi,2*pi)-pi).^2);
axis([-pi,3*pi,-1,10]);

% loop over number of terms
Ns = [1,2,5,10,15];
cols={'g','m','c','r','k'};
for iN = 1:length(Ns),
 N=Ns(iN);

 % compute partial Fourier series
 s=pi^2/3*ones(size(x));
 for n=1:N,
  an = 4*(-1)^n/n^2;
  s=s+an*cos(n*x);
 end;

 % comparative plot
 plot(x,s,cols{iN});
end;%N
hold off;
xlabel('x');
filename='p2_2_9.eps';
print('-depsc2',filename);
system(['epstopdf ' filename]);
```

```
% MATH 3150 Fall 2008
% Problem 2.2.11

thickLines(3); % remove if you don't have this in
    your system
figure(1); clf;
x = linspace(-pi,3*pi,1000);

% plot true function for refernce
hold on;
plot(x,sin(x).^2);
axis([-pi,3*pi,0,1]);

% loop over number of terms
plot(x,1/2*ones(size(x)),'r');
plot(x,1/2*(1-cos(2*x)),'g');
filename = 'p2_2_11.eps';
xlabel('x');
print('-depsc2',filename);
system(['epstopdf ',filename]);
```

```
% MATH 3150 Fall 2008
% Problem 2.2.13

thickLines(3); % remove if you don't have it in
    your system
figure(1); clf;
x = linspace(-pi,3*pi,1000);

% plot true function for refernce
hold on;
% trick to make 2*pi periodic function from fn
    def on [0,2*pi]
plot(x,(mod(x+pi,2*pi)-pi));
axis([-pi,3*pi,-pi*1.2,pi*1.2]);

% loop over number of terms
Ns = [1,2,5,10,15];
cols={'g','m','c','r','k'};
for iN = 1:length(Ns),
 N=Ns(iN);

 % compute partial Fourier series
 s=zeros(size(x));
 for n=1:N,
  an = -2*(-1)^n/n;
  s=s+an*sin(n*x);
 end;

 % comparative plot
 plot(x,s,cols{iN});
end;%N
hold off;
xlabel('x');
filename='p2_2_13.eps';
print('-depsc2',filename);
system(['epstopdf ' filename]);
```