MATH 3150 HW 1

1.1.1
$$\frac{\partial}{\partial t}(c_1u_1+c_2u_2)+\frac{\partial}{\partial x}(c_1u_1+c_2u_2)=c_1\left(\frac{\partial}{\partial t}u_1+\frac{\partial u_1}{\partial x}\right)+c_2\left(\frac{\partial u_2}{\partial t}+\frac{\partial u_2}{\partial x}\right)$$
$$=0$$
since $\frac{\partial u_i}{\partial t}+\frac{\partial u_i}{\partial x}=0,\ i=1,2.$

1.1.11 (a) We use method of characteristics to find sol. to:
$$\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial y}=0$$
characteristics satisfy $\frac{dy}{dx}=\frac{x^2}{1}\Rightarrow y=\frac{x^3}{3}+c$
$$\Rightarrow u(x,y)=f\left(y-\frac{x^3}{3}\right)$$
(b)
$$\frac{\partial u}{\partial x}=-x^2f'\left(y-\frac{x^3}{3}\right)$$
$$+\quad x^2\frac{\partial u}{\partial y}=x^2f'\left(y-\frac{x^3}{3}\right)$$
$$0=0$$

1.2.4 (a)
$$\alpha=x+ct$$
$$\beta=x-ct$$
we have:
$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}$$
$$=c\left(\frac{\partial u}{\partial \alpha}-\frac{\partial u}{\partial \beta}\right)=v$$
$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial v}{\partial t}=\frac{\partial v}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial v}{\partial \beta}\frac{\partial \beta}{\partial t}$$
$$=c^2\left(\frac{\partial^2 u}{\partial \alpha^2}-\frac{\partial^2 u}{\partial\alpha\partial\beta}\right)-c^2\left(\frac{\partial^2 u}{\partial\alpha\partial\beta}-\frac{\partial^2 u}{\partial\beta^2}\right)$$
$$=c^2\left(\frac{\partial^2 u}{\partial\alpha^2}+\frac{\partial^2 u}{\partial\beta^2}-2\frac{\partial^2 u}{\partial\alpha\partial\beta}\right)$$
similarly: $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \alpha}+\frac{\partial u}{\partial \beta}=w$
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial \alpha}+\frac{\partial w}{\partial \beta}=\frac{\partial^2 u}{\partial\alpha^2}+\frac{\partial^2 u}{\partial\beta^2}+2\frac{\partial^2 u}{\partial\alpha\partial\beta}$$
1DWEQ: $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}\iff 4c^2\frac{\partial^2 u}{\partial\alpha\partial\beta}=0\iff\frac{\partial^2 u}{\partial\alpha\partial\beta}=0$

(b) $\displaystyle\int \frac{\partial^2 u}{\partial \alpha \partial \beta}\, d\alpha = \frac{\partial u}{\partial \beta} = $ const indep of α (but may depend on β)

$= g(\beta)$

(c) $\displaystyle\int \frac{\partial u}{\partial \beta}\, d\beta = \int g(\beta)\, d\beta$

$\Rightarrow u(\alpha, \beta) = G(\beta) + $ const indep of β $= G(\beta) + F(\alpha)$
" dep of α

(d) $u(x,t) = F(x+ct) + G(x-ct)$ $(*)$

$\boxed{1.2.5}$ (a) Solve $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial u}{\partial x} \\ u(x,0) = \dfrac{1}{1+x^2} \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$ $-\infty < x < \infty$

from $(*)$: $u(x,0) = F(x) + G(x) = \dfrac{1}{x^2+1}$

$\dfrac{\partial u}{\partial t}(x,t) = cF'(x+ct) - cG'(x-ct)$

$\Rightarrow \dfrac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = 0$

$\Rightarrow F'(x) = G'(x)$

$\Rightarrow G(x) = F(x) + d$ — some constant

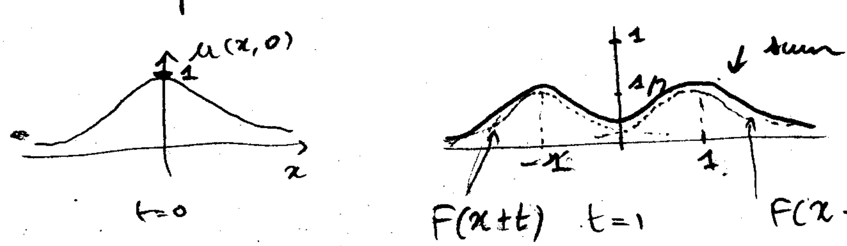
Thus: $2F(x) + d = \dfrac{1}{x^2+1}$

$\Rightarrow \begin{cases} F(x) = \dfrac{1}{2(x^2+1)} - \dfrac{d}{2} \\ G(x) = \dfrac{1}{2(x^2+1)} + \dfrac{d}{2} \end{cases}$

constant terms cancel out

$\Rightarrow u(x,t) = \dfrac{1}{2}\left[\dfrac{1}{(x+ct)^2+1} + \dfrac{1}{(x-ct)^2+1} \right]$

Note: this is a blob propagating in both + and - directions with speed c. Sketch $c=1$: (NOT REQUIRED)



$\boxed{1.2.7}$ $u(x,0)=0$

$$\frac{\partial u}{\partial t}(x,0)=-2xe^{-x^2}$$

from previous exercise we see that:

$$u(x,0)=F(x)+G(x)=0 \quad \Rightarrow F(x)=-G(x)$$

$$\frac{\partial u}{\partial t}(x,0)=cF'(x)-cG'(x)=-2xe^{-x^2}$$

$$\Rightarrow c2F'(x)=-2xe^{-x^2}$$

$$F'(x)=-\frac{2x}{2c}e^{-x^2}$$

$$F(x)=\frac{e^{-x^2}}{2c}+d$$

(some constant dep of x. however this constant is irrelevant for final solution as it cancels out)

$$G(x)=-\frac{e^{-x^2}}{2c}-d$$

$$\Rightarrow \boxed{u(x,t)=\frac{1}{2c}\left[e^{-(x+ct)^2}-e^{-(x-ct)^2}\right]}$$

$\boxed{1.2.16}$ we know that $u_n(x,t)=\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$ solves 1DWEQ with $u_n(x,0)=\sin\frac{n\pi x}{L}$

and $\frac{\partial u_n}{\partial t}(x,0)=0$

thus if u solves 1DWEQ with $\begin{cases} u(x,0)=f(x)=\frac{1}{2}\sin\frac{\pi x}{L}+\frac{1}{4}\sin\frac{3\pi x}{L} \\ \frac{\partial u}{\partial t}(x,0)=0 \end{cases}$

then $$\boxed{u(x,t)=\underbrace{\frac{1}{2}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}}_{u_1(x,t)}+\frac{1}{4}\underbrace{\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L}}_{u_3(x,t)}}$$

since by linearity: u solves 1DWEQ

$$u(x,0)=f(x)$$

$$\frac{\partial u}{\partial t}(x,0)=0$$