\[ \begin{align*}
\frac{\partial}{\partial t} \left( c_1 u + c_2 u^2 \right) + \frac{\partial}{\partial x} \left( c_1 u + c_2 u^2 \right) &= c_1 \left( \frac{\partial u}{\partial t} + c_2 \frac{\partial u}{\partial x} \right) + c_2 \left( \frac{\partial u}{\partial t} + c_2 \frac{\partial u}{\partial x} \right) \\
\quad &= 0
\end{align*} \]

since \( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \), \( c = c_2 \).

4.4.41 (a) We use method of characteristics to find solution:

\[ \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0 \]

characteristics satisfy \( \frac{dy}{dx} = \frac{x^2}{1} \Rightarrow y = \frac{x^3}{3} + c \)

\( u(x, y) = f \left( y - \frac{x^3}{3} \right) \)

\( b) \quad \frac{\partial u}{\partial x} = -x^2 f' \left( y - \frac{x^3}{3} \right) \)

\( + \quad x^2 \frac{\partial u}{\partial y} = x^2 f' \left( y - \frac{x^3}{3} \right) \)

\( 0 = 0 \)

4.2.4

(a) \( \alpha = x + ct \)
\( \beta = x - ct \)

we have:

\[ \begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial t} \\
\quad &= c \left( \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta} \right) = 0 \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \beta} \right) \\
\quad &= c^2 \left( \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \beta^2} \right) - c^2 \left( \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta \partial \alpha} \right) - \frac{\partial^2 u}{\partial \alpha \partial \beta} \\
\quad &= c^2 \left( \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \beta^2} + 2 \frac{\partial^2 u}{\partial \alpha \partial \beta} \right)
\end{align*} \]

Similarly:

\[ \begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} = \omega \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial \beta} \right) = \frac{\partial^2 u}{\partial \alpha \partial x} + \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta \partial x} + \frac{\partial^2 u}{\partial \beta \partial \beta}
\end{align*} \]

10.0.0: \( \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial u}{\partial \beta} \quad (\Rightarrow) \quad c^2 \frac{\partial u}{\partial \beta} = 0 \quad (\Rightarrow) \quad \frac{\partial u}{\partial \beta} = 0 \)
\[
\begin{align*}
(b) \quad \int \frac{\partial^2 u}{\partial x \partial \beta} \, d\beta &= \frac{\partial u}{\partial \beta} = \text{const ind of } x \quad \text{(but may depend on } \beta) \\
&= g(\beta) \\
(c) \quad \int \frac{\partial u}{\partial \beta} \, d\beta &= \int g(\beta) \, d\beta \\
&= u(x, \beta) + \text{const ind of } \beta = G(\beta) + F(x) \\
&= \mu(x, \beta) = g(\beta) + \text{const ind of } \beta = G(\beta) + F(x) \\
(d) \quad \mu(x, t) &= F(x + ct) + G(x - ct) \quad \text{(*)}
\end{align*}
\]

4.2.5
(a) Solve \[
\left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \\
u(x, 0) = \frac{1}{1 + x^2} \\
\frac{\partial u}{\partial t}(x, 0) = 0
\end{array} \right.
\]
from (i): \[
u(x, 0) = F(x) + G(x) = \frac{1}{x^2 + 1}
\]
\[
\frac{\partial u}{\partial t}(x, t) = c F'(x + ct) - c G'(x - ct).
\]
\[
\Rightarrow \quad \frac{\partial u}{\partial x}(x, t) = c F'(x) - c G'(x) = 0 \quad \text{some constant}
\Rightarrow \quad F'(x) = G'(x)
\Rightarrow \quad G(x) = F(x) + d
\]
Thus: \[
2F(x) + d = \frac{1}{x^2 + 1}
\]
\[
\Rightarrow \quad \left\{ \begin{array}{l}
F(x) = \frac{1}{2(x^2 + 1)} - \frac{d}{2} \\
G(x) = \frac{1}{2(x^2 + 1)} + \frac{d}{2}
\end{array} \right. \quad \text{constant terms cancel out}
\]
\[
\Rightarrow \quad \mu(x, t) = \frac{1}{2} \left[ \frac{1}{(x + ct)^2 + 1} + \frac{1}{(x - ct)^2 + 1} \right]
\]
Note: this is a shock propagating in both + and - directions with speed c. Sketch c = 1. (not required)
1.2.7 \[ \mu(x, 0) = 0 \]
\[ \frac{\partial \mu}{\partial t}(x, 0) = -2xe^{-x^2} \]

From previous exercise we see that:

\[ 2e(x, 0) = F(x) + G(x) = 0 \quad \Rightarrow \quad F(x) = -G(x) \]
\[ \frac{\partial e}{\partial t}(x, 0) = cF'(x) - cG'(x) = -2xe^{-x^2} \]

\[ \Rightarrow \quad cF'(x) = -2xe^{-x^2} \]
\[ F(x) = -\frac{2x}{2c} e^{-x^2} + c \] (same constant dep of x. however this constant is irrelevant for final solution as it cancels out)
\[ G(x) = -\frac{e^{-x^2}}{2c} - d \]

\[ \Rightarrow \quad \mu(u, t) = \frac{1}{2c} \left[ e^{-(x+ct)^2} - e^{-(x-ct)^2} \right] \]

1.2.16 We know that \( \mu(x, t) = \frac{\sin \frac{\pi t \mu}{L}}{L} \) solves 1DWEQ with \( \mu(x, 0) = \frac{\sin \frac{\pi t \mu}{L}}{L} \) and \( \frac{\partial \mu}{\partial t}(x, 0) = 0 \)

Thus if \( \mu \) solves 1DWEQ with

\[ \begin{cases} \mu(x, t) = \frac{1}{2} \sin \frac{\pi \mu}{L} \cos \frac{3\pi ct}{L} + \frac{1}{4} \sin \frac{\pi \mu}{L} \cos \frac{3\pi c t}{L} \\ \frac{\partial \mu}{\partial t}(x, 0) = 0 \end{cases} \]

then

\[ \Rightarrow \quad \mu(x, t) = \frac{1}{2} \sin \frac{\pi \mu}{L} \cos \frac{3\pi ct}{L} + \frac{1}{4} \sin \frac{\pi \mu}{L} \cos \frac{3\pi c t}{L} \]

\[ \mu(x, t) = \frac{1}{2} \sin \frac{\pi \mu}{L} \cos \frac{3\pi ct}{L} + \frac{1}{4} \sin \frac{\pi \mu}{L} \cos \frac{3\pi c t}{L} \]

Since by linearity: \( \mu \) solves 1DWEQ

\[ \begin{cases} \mu(x, 0) = \frac{1}{2} \sin \frac{\pi \mu}{L} \\ \frac{\partial \mu}{\partial t}(x, 0) = 0 \end{cases} \]