§ 7.9 Nonhomogeneous Heat Equation

Welding torch (Example 7.9.1)

$$(1)\quad \begin{cases} u_t = c^2 u_{xx} & x \in \mathbb{R},\ t>0 \\ u(x,0) = \delta_0(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = g_t * \delta_0(x) = \frac{1}{\sqrt{2\pi}} g_t(x) = \frac{1}{2c\sqrt{\pi t}} e^{-x^2/(4c^2t)}$$

where $\boxed{g_t(x) = \frac{1}{c\sqrt{2t}} e^{-x^2/4c^2t} = \text{heat kernel}}$

little problem: $g_t(x)$ is only defined for $t>0$, but we have an initial condition $(t=0)$ ...

in fact initial condition holds in the sense of distributions.

$$\boxed{\lim_{t\to 0} \langle u(x,t), f(x)\rangle = \langle \delta_0, f(x)\rangle}$$

This is easy to check:

$$\lim_{t\to 0} \langle u(x,t), f(x)\rangle = \lim_{t\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_t(x) f(x)\,dx$$

g_t even

$$= \lim_{t\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_t(0-x) f(x)\,dx$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{2\pi}} (g_t * f)(0) \underset{\text{theorem 7.4.1}}{=} f(0)$$

$u(x,t)$ is called a <u>weak solution</u> of (1) because it satisfies (1) in the distribution sense / gen. fun.

However we will simply call them "solution"

Fundamental solution of the Heat Equation

Let
$$\phi(x,t) = \begin{cases} \dfrac{1}{2c\sqrt{\pi t}}\, e^{-x^2/(4c^2t)} & \text{if } t>0 \\ 0 & \text{if } t\leq 0 \end{cases}$$

$$\hat{\phi}(\omega,t) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\, e^{-c^2\omega^2 t} & \text{if } t>0 \\ 0 & \text{if } t\leq 0 \end{cases}$$

Theorem $\phi(x,t)$ is a weak solution to the non homog heat eq:

$$u_t = c^2 u_{xx} + \delta_0(x)\,\delta_0(t) \quad , \quad t, x \in \mathbb{R}$$

ϕ is called fundamental sol or Green's function and it can be used to recover solutions for arbitrary nonhomogeneous heat eq.

proof: it is easy to see ϕ is a sol to $u_t = c^2 u_{xx}$ (homog heat eq.)

To prove theorem take $\mathcal{F}$ on both sides:

$$\frac{d}{dt}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t) + \frac{1}{\sqrt{2\pi}}\,\delta_0(t)$$

$$\frac{d}{dt}\hat{u}(\omega,t) + c^2\omega^2\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\,\delta_0(t) \qquad (2)$$

now we must show that $\hat{\phi}$ is a weak solution to (2) i.e.:

$$\left\langle \frac{d}{dt}\hat{\phi}(\omega,t) + c^2\omega^2\hat{\phi}(\omega,t), f\right\rangle_t = \frac{1}{\sqrt{2\pi}}\langle \delta_0, f\rangle_t = f(0)$$

$$\parallel$$

$$-\langle\hat{\phi}, f'\rangle + c^2\omega^2\langle\hat{\phi}, f\rangle$$

$$= -\int_0^\infty \hat{\phi}(\omega,t)f'(t)\,dt + c^2\omega^2\int_0^\infty \hat{\phi}(\omega,t)f(t)\,dt$$

$t>0$:

$$\hat{\phi}(\omega,t) = \frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}$$

$$\frac{d}{dt}\hat{\phi}(\omega,t) = -\frac{1}{\sqrt{2\pi}} c^2\omega^2 e^{-c^2\omega^2 t}$$

$$\Rightarrow -\int_0^\infty \hat{\phi}(\omega,t) f'(t)\,dt = -\frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t} f(t)\Big|_0^\infty - \frac{1}{\sqrt{2\pi}} c^2\omega^2 \int_0^\infty e^{-c^2\omega^2 t} f(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}} f(0) - \frac{1}{\sqrt{2\pi}} c^2\omega^2 \int_0^\infty e^{-c^2\omega^2 t} f(t)\,dt$$

$\Rightarrow$ get result.

Theorem: The sol to:

$$\begin{cases} u_t = c^2 u_{xx} + f(x,t) & (t>0,\ x\in\mathbb{R}) \\ u(x,0) = 0 \end{cases}$$

is given by:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \phi(x-y,t-s) f(y,s)\,dy\,ds$$

where $\phi(x,t) = \frac{1}{\sqrt{2\pi}} g_t(x) = \frac{1}{2c\sqrt{\pi t}} e^{-x^2/(4c^2t)}$ if $t>0$ and $x\in\mathbb{R}$

This solves heat equation with arbitrary source term.