Thus
$$\boxed{A_j = \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a f(r)\, J_p\left(\frac{\alpha_{pj}}{a}\right) \mathbf{r}\, dr}$$

do not forget that this inner product is WEIGHTED.

Recall that after using polar coordinates and using symmetries of problem, we get the simplified PDE that describes vibrations of a membrane with radially symmetric initial conditions:

$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right) \\ u(a,t) = 0 \\ u(r,0) = f(r) \\ u_t(r,0) = g(r) \end{cases}$$

Using separation of variables we reach the solution:

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos c\lambda_n t + B_n \sin c\lambda_n t\right) J_0(\lambda_n r)$$

where $J_0(\cdot)$ = Bessel function of order 0 of the first kind

$\lambda_n = \frac{\alpha_n}{a}$, α_n = n-th zero of J_0 (i.e. $J_0(\alpha_n) = 0$)

The A_n and B_n come from initial pos and init vel resp.

$$A_n = \frac{(f(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))} = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r)\, r\, dr$$

$$c\lambda_n B_n = \frac{(g(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))} = \text{"} \int_0^a g(r) J_0(\lambda_n r)\, r\, dr$$

Here is a particular example (exampl 4.2.2)

$$f(r) = 1 - r^2$$
$$g(r) = 0 \qquad \Rightarrow B_n = 0$$

$$\boxed{A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1-r^2) J_0(\alpha_n r)\, r\, dr}$$

$s = \alpha_n r$
$ds = \alpha_n dr$

$$= \frac{2}{J_1^2(\alpha_n)} \int_0^{\alpha_n} \left(1 - \left(\frac{s}{\alpha_n}\right)^2\right) J_0(s) \frac{s}{\alpha_n} \frac{ds}{\alpha_n}$$

$$= \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s)\, s\, ds$$

IBP:
$(\alpha_n^2 - s^2)' = -2s$
$\int J_0(s)\, s\, ds \doteq J_1(s)\, s$
(see p47 of notes)

$$= \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \left[(\alpha_n^2 - s^2) J_1(s)\, s \Big|_0^{\alpha_n} + 2 \int_0^{\alpha_n} J_1(s) s^2 ds \right]$$

$$= \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_1(s) s^2 ds$$

again from p47 in notes

$$= \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} J_2(s) s^2 \Big|_0^{\alpha_n} = \boxed{\frac{4}{\alpha_n^2 J_1^2(\alpha_n)} J_2(\alpha_n)}$$

Show Matlab version

§4.3 Vibrations of circular membrane (General case)

This time we do not assume I.C. are radially symmetric. We want to solve:

$$\begin{cases} u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) = c^2 \Delta u \quad \text{(in polar coord)} \\ u(r,\theta,0) = f(r,\theta) \\ u_t(r,\theta,0) = g(r,\theta) \\ u(a,\theta,t) = 0 \end{cases}$$

} Initial conditions

} Boundary conditions

Solution using separation of variable: $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda^2 = \text{constant}$$

(sign to get periodic sol. in T, which is what physics tells us)

(1) $\dfrac{T''}{c^2T} = -\lambda^2$ and $\dfrac{R''}{R} + \dfrac{R'}{rR} + \dfrac{\Theta''}{r^2\Theta} = -\lambda^2$

Separation of variables again

(2) $\lambda^2 r^2 + \dfrac{r^2R''}{R} + r\dfrac{R'}{R} = \mu^2$

(3) $-\dfrac{\Theta''}{\Theta} = \mu^2$

(sign because we know $u(r,\theta,t)$ should be 2π-per in θ: $u(r,\theta+2\pi,t) = u(r,\theta,t)$)

(3): Using periodicity of Θ and Θ':

$$\begin{cases} \Theta'' + \mu^2\Theta = 0 \\ \Theta(0) = \Theta(2\pi) = 0 \\ \Theta'(0) = \Theta(2\pi) = 0 \end{cases} \Rightarrow \Theta_m = A_m \cos m\theta + B_m \sin m\theta$$

for $m = 0, 1, 2, \dots$

(2):
$$\begin{cases} r^2R'' + rR' + (\lambda^2r^2 - m^2)R = 0 \\ R(a) = 0 \end{cases}$$

We saw that $J_m(x)$, $Y_m(x)$ solve:

$x^2y'' + xy' + (x^2 - m^2)y = 0$, which is almost like (2).

Letting $x = \lambda r$ and $R(r) = y(\lambda r)$,

$\Rightarrow R'(r) = \lambda y'(\lambda r)$

$R''(r) = \lambda^2 y''(\lambda r)$

$$\Rightarrow (\lambda r)^2 \frac{R''}{\cancel{\lambda^2}} + \lambda r \frac{R'}{\cancel{\lambda}} + (\lambda^2r^2 - m^2)R = 0 \text{ which is (2)}$$

Therefore:

$$R(r) = c_1 J_m(\lambda r) + \cancel{c_2 Y_m(\lambda r)}$$

$c_2 = 0$ because $Y_m(r) \to -\infty$ as $r \to 0$.

$$\Rightarrow \quad R(r) = J_m(\lambda r)$$

using boundary conditions:

$$R(a) = J_m(\lambda a) = 0 \Rightarrow \lambda_{mn} = \frac{\alpha_{mn}}{a} \quad \left(\begin{matrix} m = 0, 1, 2 \dots \\ n = 1, 2, 3 \dots \end{matrix} \right)$$

where α_{mn} is the nth zero of $J_m(r)$.

$$\Rightarrow \boxed{R_{mn}(r) = J_m(\lambda_{mn} r)}$$

Now (1) gives solutions of the form $a \cdot \cos c\lambda_{mn} t + b \sin c\lambda_{mn} t$

Thus the solution has the form:

$$u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn} \cos m\theta + b_{mn} \sin m\theta\right) \cos c\lambda_{mn} t$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a^*_{mn} \cos m\theta + b^*_{mn} \sin m\theta\right) \sin c\lambda_{mn} t$$

Here a_{mn}, b_{mn} correspond to $f(r,\theta)$

and a^*_{mn}, b^*_{mn} — $g(r,\theta)$.

How to get these? a_{mn} and b_{mn}: (*)

$$u(r,\theta,0) = \boxed{f(r,\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn} \cos m\theta + b_{mn} \sin m\theta\right)}$$

~ generalized Fourier series for f

To get coefficients, freeze r and look at $(*)$ as if it where a Fourier series:

$$f(r,\theta) = \underbrace{\sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)}_{= a_0(r)}$$

$$+ \sum_{m=1}^{\infty} \left[\underbrace{\left\{ \sum_{n=1}^{\infty} a_{mn} J_m(\lambda_{mn} r) \right\}}_{= a_m(r)} \cos m\theta + \underbrace{\left\{ \sum_{n=1}^{\infty} b_{mn} J_m(\lambda_{mn} r) \right\}}_{= b_m(r)} \sin m\theta \right]$$

$$= a_0(r) + \sum_{m=1}^{\infty} a_m(r) \cos m\theta + b_m(r) \sin m\theta$$

$$\Rightarrow \quad a_0(r) = \frac{(f(r,\theta), 1)}{(1, 1)} = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta)\, d\theta$$

$$a_m(r) = \frac{(f(r,\theta), \cos m\theta)}{(\cos m\theta, \cos m\theta)} = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \cos m\theta\, d\theta$$

$$b_m(r) = \frac{(f(r,\theta), \sin m\theta)}{(\sin m\theta, \sin m\theta)} = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \sin m\theta\, d\theta$$

But each $a_m(r)$ or $b_m(r)$ is a Bessel series thus:

$$a_{mn} = \frac{(a_0(r), J_m(\lambda_{mn} r))}{(J_m(\lambda_{mn} r), J_m(\lambda_{mn} r))}$$

$$= \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a a_m(r) J_m(\lambda_{mn} r)\, r\, dr$$

$$= \begin{cases} \dfrac{1}{2\pi} \dfrac{2}{a^2 J_1^2(\alpha_{0n})} \displaystyle\int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta) J_0(\lambda_{0n} r) & (m=0) \\[2ex] \dfrac{1}{\pi} \dfrac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \displaystyle\int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta) J_m(\lambda_{mn} r) \cos m\theta & (m>0) \end{cases}$$

And similarly:

$$b_{mn} = \frac{1}{\pi} \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta)\, J_m(\lambda_{m,n} r) \sin m\theta$$

$m = 1, 2, \ldots$

$n = 1, 2, \ldots$

It's a good exercise to check that: (w/ similar reasoning)

$$a_{0n}^* = \frac{1}{c\alpha_{0n}} \frac{1}{2\pi} \frac{2}{a^2 J_1^2(\alpha_{0n})} \int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta)\, J_0(\lambda_{0n} r)$$

$$a_{mn}^* = \frac{1}{c\alpha_{mn}} \frac{1}{\pi} \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta)\, J_m(\lambda_{mn} r) \cos m\theta$$

$$b_{mn}^* = \frac{1}{c\alpha_{mn}} \frac{1}{\pi} \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta)\, J_m(\lambda_{mn} r) \sin m\theta$$

See book's examples

§4.4 Laplace's Equation in circular regions

$$\begin{cases} \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u(a,\theta) = f(\theta) \qquad 0 < \theta < 2\pi \end{cases}$$

Laplace eq. w/ Dirichlet B.C.

Separation of variables:

$u(r,\theta) = R(r)\Theta(\theta)$ plugging into DE we get:

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

sign chosen s.t. Θ is periodic

$$\Theta'' + m\Theta = 0 \Rightarrow \Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \quad (n = 0, 1, 2, \ldots$$

$$r^2 R'' + r R' - n^2 R = 0$$

App. A3 or Variation of param. method?

$$\begin{cases} R(r) = c_1 \left(\frac{r}{a}\right)^n + c_2 \left(\frac{r}{a}\right)^{-n} \\ R(r) = c_1 + c_2 \ln\left(\frac{r}{a}\right) \end{cases}$$

Now $c_2 = 0$ since we do not want unbounded solutions for $r=0$.

thus:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right].$$

$$\Rightarrow u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$