**Polar coordinates**

\[
\begin{align*}
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta
\end{cases}
\end{align*}
\]

\[r^2 = x^2 + y^2, \quad \tan \theta = \frac{x}{y}\]

\[
\Theta = \tan^{-1} \frac{y}{x} + k \pi,
\]

where \(k\) depends on \(x\) and \(y\).

\[
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}
\]

(all terms have same units as \(r^{-2}\)).

The representation of Laplacian in polar coordinates.

Deriving is long and tedious but it depends only on using chain rule and knowing derivatives makes:

\[
\begin{align*}
\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2 + y^2}} = \cos \theta \\
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2 + y^2}} = \sin \theta \\
\frac{\partial \theta}{\partial x} &= \frac{1}{1 + (y/x)^2} \left( -\frac{y}{x^2} \right) = -\frac{y}{r^2} = -\frac{\sin \theta}{r} \\
\frac{\partial \theta}{\partial y} &= \frac{1}{1 + (y/x)^2} \left( \frac{x}{x^2} \right) = \frac{x}{r^2} = \frac{\cos \theta}{r}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{x}{\sqrt{x^2 + y^2}} \right) = \frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \frac{1}{r^2} \\
\frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{y}{\sqrt{x^2 + y^2}} \right) = \frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \frac{1}{r^2} \\
\frac{\partial^2 \theta}{\partial x^2} &= \frac{1}{r^2} \left( \frac{\sin \theta}{r} \right) = -\frac{1}{r^2} \left( \cos \theta \right) \frac{1}{r} = -\frac{\cos \theta}{r^2} \\
\frac{\partial^2 \theta}{\partial y^2} &= \frac{1}{r^2} \left( \frac{\cos \theta}{r} \right) = \frac{\cos \theta}{r^2} + \frac{\sin \theta}{r} \frac{1}{r} = \frac{\cos \theta}{r^2} + \frac{\sin \theta}{r}
\end{align*}
\]

These are included to practice with changes of coordinates, see p.195 for more details.
Cylindrical coordinates

\[
\begin{align*}
\alpha &= r \cos \phi \\
g &= r \sin \phi \\
\phi &= \phi
\end{align*}
\]

In cartesian coordinates:

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\]

Using polar form of Laplacian:

\[
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial \phi^2}
\]

Spherical coordinates

\[
\begin{align*}
x &= r \sin \phi \cos \theta \\
y &= r \sin \phi \sin \theta \\
z &= r \cos \phi
\end{align*}
\]

We have:

\[
r^2 = x^2 + y^2 + z^2
\]

\[
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\sin \phi} \frac{\partial u}{\partial \phi} + \frac{1}{\sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2} \right)
\]

Derivation involves using Laplacian polar form on \(xy\)-plane and \(\phi\) plane, where \(\phi\) is projection of \((x,y,z)\) onto \((x,y)\) plane. See p. 137. We are not going to cover \(\phi\) so we won't use this too much in this class.
§ 4.2 Vibrations of a circular membrane (radially symm)

We study vibrations of a circular membrane (drum) clamped on edge i.e. \( \mu(\alpha, \Theta, t) = 0 \), where \( \mu(r, \Theta, t) \) = displacement of membrane from equilibrium.

These are governed by the 2D wave equation:

\[
\frac{\partial^2 \mu}{\partial t^2} = c^2 \Delta \mu
\]

which in polar coordinates (better adapted to this problem than cartesians)

\[
\frac{\partial^2 \mu}{\partial t^2} = c^2 \left( \frac{\partial^2 \mu}{\partial r^2} + \frac{1}{r} \frac{\partial \mu}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \mu}{\partial \Theta^2} \right)
\]

We consider first the case where:

- initial shape \( f(r, \Theta) = f(r) \) \hspace{1cm} (radially symm)
- initial velocity \( g(r, \Theta) = g(r) \)

Because of the symmetry of the problem, \( \mu(r, \Theta) = \mu(r) \)

\[
\Rightarrow \frac{\partial \mu}{\partial \Theta} = 0
\]

Thus we are left with the simplified PDE:

\[
\frac{\partial^2 \mu}{\partial t^2} = c^2 \left( \frac{\partial^2 \mu}{\partial r^2} + \frac{1}{r} \frac{\partial \mu}{\partial r} \right)
\]

\[
\mu(\alpha, t) = 0
\]

\[
\mu(r, 0) = f(r)
\]

\[
\frac{\partial \mu}{\partial t}(r, 0) = g(r)
\]

Use separation of variables: \( \mu(r, t) = R(r)T(t) \)

\[
RT'' = c^2 \left( R''T + \frac{1}{r} R'T \right)
\]

\[
\Rightarrow \frac{T''}{c^2 T} = -\frac{1}{r} \left( R'' + \frac{1}{r} R' \right) = -\frac{\lambda}{c^2} = \text{const}
\]

or have no time periodic sol!
We get 2 equations to solve:

\( \begin{cases} R'' + R' + \lambda r R = 0 \\ R(0) = 0 \end{cases} \)  \( \lambda \) is constant.

(1) Bessel equation of order 0 (parametric form)

2nd order linear eq. we need 2 lin. indep. sol to
determine all possible solutions.

\[ J_0(\lambda r) \]  Bessel function of order 0 of first kind
\[ Y_0(\lambda r) \]  2nd

Solving (1) we get:

\[ R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r) \]

\[ r \to 0 \Rightarrow Y_0(\lambda r) \to \infty \]

Since \(|R(0)| < \infty\)
we must have \(c_2 = 0\)

\[ R(r) = c_1 J_0(\lambda r) \]
\[ R(0) = c_1 J_0(\lambda a) = 0 \]

\( \lambda a \) must be a root of Bessel func. \( J_0 \).

\[ \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots \]  be zeros of \( J_0 \)

then \[ \lambda = \alpha_n = \frac{\alpha}{a} \]

and \[ R_n(r) = J_0\left(\frac{\alpha_n}{a} r\right), \quad n = 1, 2, \ldots \]

(analogous to the zeros in the sin series expansion)

we solve \[ \sin \left( \frac{n \pi}{L} \right) = 0 \]  \( n = \frac{m \pi}{L} \)

and analogy does not stop here!
They are similar to (2) and write its solution:

\[ T_n(t) = A_n \cos \left( \frac{\tan r}{a} \right) + B_n \sin \left( \frac{\tan r}{a} \right) \]

By construction:

\[ u_n(r,t) = (A_n \cos \left( \frac{\tan r}{a} \right) + B_n \sin \left( \frac{\tan r}{a} \right)) J_0 \left( \frac{\tan r}{a} \right) \]

\[ n = 1, 2, \ldots \]

solves DE and so does:

\[ M(r,t) = \sum_{n=1}^{\infty} (A_n \cos (c_n t) + B_n \sin (c_n t)) J_0 (c_n r) \]

What about B.C.? \[ f(r) = M(r,0) = \sum_{n=1}^{\infty} A_n J_0 (c_n r) \] Bessel Series Expansion

Looks a lot like Fourier series expansion, and it relies on \( J_0 (c_n r) \) being orthogonal in the inner product:

\[ \langle u, v \rangle = \int_0^a u(r) v(r) r dr \quad \text{compare to} \quad \langle u, v \rangle = \int_0^\infty u(r) v(r) \rho dr \]

and:

\[ \langle J_0 (c_n r), J_0 (c_m r) \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \frac{a^2 J_1^2(c_n)}{2} & \text{if } n = m \end{cases} \]

Thus:

\[ f(r) = \sum_{n=1}^{\infty} A_n J_0 (c_n r) \]

Thus:

\[ A_n = \frac{\langle f(r), J_0 (c_n r) \rangle}{\langle J_0 (c_n r), J_0 (c_n r) \rangle} = \frac{2}{a^2 J_1^2(c_n)} \int_0^a f(r) J_0 (c_n r) r dr \]

Here \( J_1(r) = \text{Bessel fun of order } 1 \) (available in most...