§ 3.8 Laplace Equation in rectangular coordinates

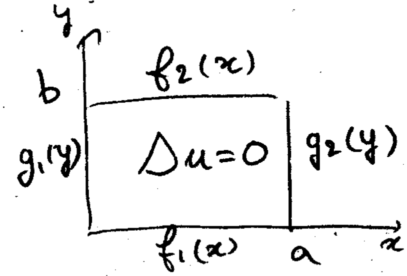
Recall heat eq. in 2D: $u_t = c^2(u_{xx} + u_{yy}) = c^2 \Delta u$ Laplacian

Steady state $u_t = 0 \Rightarrow \Delta u = 0$ Laplace equation

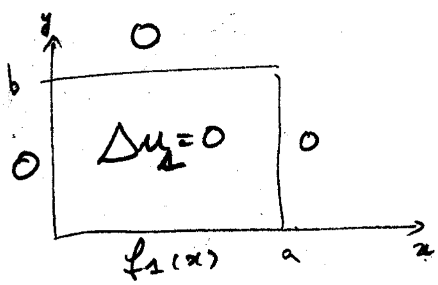
Solving Laplace equation with separation of variables

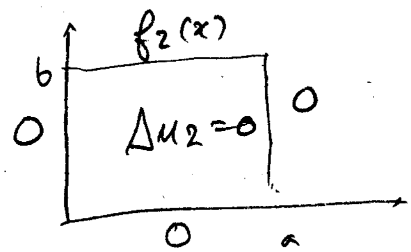
The full blown problem is:

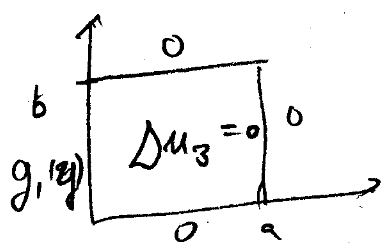
$$(*) \begin{cases} \Delta u = 0 \\ u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = g_1(y) \\ u(a,y) = g_2(y) \end{cases}$$

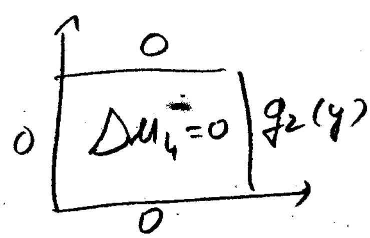


Simplify problem by using linearity: solve for u_1, u_2, u_3, u_4









by linearity the sol to $(*)$ is $u = u_1 + u_2 + u_3 + u_4$

Let us solve for u_1 (cf. example 3.8.1 for u_2) using sep of variables:

$$u_1(x,y) = X(x)Y(y)$$

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = k \text{ constant.}$$

We get:

$$(PX)\begin{cases} X'' + kX = 0 \\ X(0) = X(a) = 0 \end{cases} \qquad (PY)\begin{cases} Y'' - kY = 0 \\ Y(b) = 0 \end{cases}$$

(PX): $k \leq 0$ leads to $X(x) = 0$ (trivial solution)

let $k = +\mu^2 \Rightarrow X(x) = A\cos\mu x + B\sin\mu x$

$$X(0) = A = 0$$

$$X(a) = B\sin\mu a = 0$$

$$\Rightarrow \mu = \mu_n = \frac{n\pi}{a}, \quad n = 1, 2, \dots$$

$$\Rightarrow \boxed{X_n(x) = \sin\frac{n\pi}{a}x}, \quad n = 1, 2 \dots$$

(PY): $Y_n(y) = A'_n\cosh\mu_n y + B'_n\sinh\mu_n y$

B.C.: $Y_n(b) = 0 \Rightarrow A'_n\cosh\mu_n b + B'_n\sinh\mu_n b = 0$

$$\Rightarrow \frac{\sinh\mu_n b}{\cosh\mu_n b} = \tanh\mu_n b = -\frac{A'_n}{B'_n}$$

$$\Rightarrow A'_n = A_n\sinh\mu_n b = A_n\sinh\frac{n\pi}{a}b$$

$$B'_n = -A_n\cosh\mu_n b = -A_n\cosh\frac{n\pi}{a}b$$

$$\Rightarrow \boxed{Y_n(y) = A_n\left[\sinh\mu_n b\cosh\mu_n y - \cosh\mu_n b\sinh\mu_n y\right]}$$

$$\boxed{= A_n\sinh(\mu_n(b-y))}$$

Recall: $\sinh(a+b) = \sinh(a)\cosh b + \sinh b\cosh a$

Thus $u_n(x,y) = X_n(x)Y_n(y) = A_n\sin\frac{n\pi}{a}x\,\sinh(\mu_n(b-y))$

solves PDE by construction, and so does

$$\boxed{u_1(x,y) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{a}x\,\sinh(\mu_n(b-y))}$$

Now use boundary condition :

$$u_1(0,x) = f_1(x)$$

$$= \sum_{n=1}^{\infty} \underbrace{A_n \sinh \frac{n\pi}{a} b}_{\text{Sine coeff of } f_1(x)} \sin \frac{n\pi}{a} x$$

$$\Rightarrow \boxed{A_n = \frac{2}{a} \frac{1}{\sinh \frac{n\pi}{a} b} \int_0^a f_1(x) \sin \frac{n\pi}{a} x \, dx}$$

We can do this similarly with the other B.C.

The solution to the problem w/ arbitrary Dirichlet B.C. is:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} (b-y)$$
$$+ B_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$
$$+ C_n \sinh \frac{n\pi}{b} (a-x) \sin \frac{n\pi}{b} y$$
$$+ D_n \sinh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y$$

where

$A_n \sinh \frac{n\pi}{a} b$ = Sine series coeff of $f_1(x)$

$B_n \sinh \frac{n\pi}{a} b$ = ———————— $f_2(x)$

$C_n \sinh \frac{n\pi}{b} a$ = ———————— $g_1(y)$

$D_n \sinh \frac{n\pi}{b} a$ = ———————— $g_2(y)$.