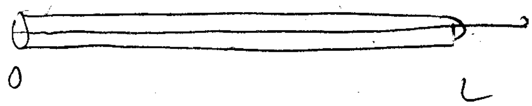
§3.6 Even more boundary conditions for 1D Heat Eq



metal rod with <u>insulated ends</u>

homogeneous Neumann B. C.

$$(1)\begin{cases} u_t = c^2 u_{xx}, & 0<x<L,\ t>0 \\ u_x(0,t) = u_x(L,t) = 0, & t>0 \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

$\Leftrightarrow$ no heat flux

<u>Solution using separation of variable.</u>

$u(x,t) = X(x)T(t)$ = ansatz plug in in (1) gives:

$$\begin{cases} X'' - kX = 0 \\ X'(0) = X'(L) = 0 \end{cases} \qquad T' - kc^2 T = 0$$

$\underline{k = \mu^2 > 0}$: $X(x) = a\cosh\mu x + b\sinh\mu x$ + B.C. $\Rightarrow$ $X(x) = 0$ (Check)

$\underline{k = 0}$: $X(x) = ax + b$ + B.C.: $X'(x) = a$

$X'(0) = X'(L) = 0 \Rightarrow a = 0$

$\Rightarrow$ we get $X(x) = b$ a nontrivial solution

$\underline{k = -\mu^2 < 0}$: $X(x) = a\cos\mu x + b\sin\mu x$

$X'(x) = -a\mu\sin\mu x + b\mu\cos\mu x$

$X'(0) = 0 \Rightarrow b = 0$

$X'(L) = 0 \Rightarrow a\sin\mu L = 0$

$\Rightarrow \mu = \mu_n = \frac{n\pi}{L}$, $n = 1, 2, \ldots$

$\rightsquigarrow X_n(x) = \cos\frac{n\pi x}{L}$, $n = 1, 2, \ldots$

Thus we get:

$T_0(t) = a_0$

$T_n(t) = a_n \exp\left[-\left(\frac{n\pi c}{L}\right)^2 t\right]$

Using superposition principle:

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-\left(\frac{n\pi c}{L}\right)^2 t\right] \cos\left(\frac{n\pi}{L}x\right)}$$

solves by construction (4)?

What about initial conditions?

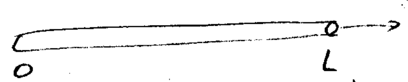
$$u(x,0) = f(x) = a_0 + \sum_{n=1} a_n \cos\left(\frac{n\pi}{L}x\right)$$

surprise! a_n, $n \geq 0$ are the coeff in the cosine series of $f(x)$.

$$\Rightarrow \quad a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi}{L}x \, dx$$

Now more complicated Robin type boundary cdts:



heat transfer / convection coefficient

Dirichlet $u(0,t)=0$ (ice bath)

Robin $u_x(L,t) = -\mathcal{K} u(L,t)$ Bar loses heat at a rate proportional to temperature of medium.

$$(2)\quad \begin{cases} u_t = c^2 u_{xx}, & 0<x<L,\ t>0 \\ u_x(0,t) = 0, & t>0 \\ u_x(L,t) = -\mathcal{K}u(L,t), & t>0 \\ u(x,0) = f(x), & 0<x<L \end{cases}$$

Idea Use separation of variables to get:

$$\begin{cases} X'' - kX = 0 \\ X(0) = 0 & (BC1) \\ X'(L) = -\mathcal{K}X(L) & (BC2) \end{cases} \qquad T' - kc^2 T = 0$$

$k = \mu^2 > 0$: $X(x) = a\cosh\mu x + b\sinh\mu x$. (BC1) $\Rightarrow a = 0$

(BC2) $\Rightarrow b \underbrace{\sinh\mu L}_{>0} = 0 \Rightarrow b = 0$

$\Rightarrow X(x) = 0$ (trivial sol)

$k=0$ $X(x) = ax+b$. (BC1) $\Rightarrow b = 0$

(BC2) $\Rightarrow a = -\mathcal{K}aL \Rightarrow a(1+\mathcal{K}L) = 0$

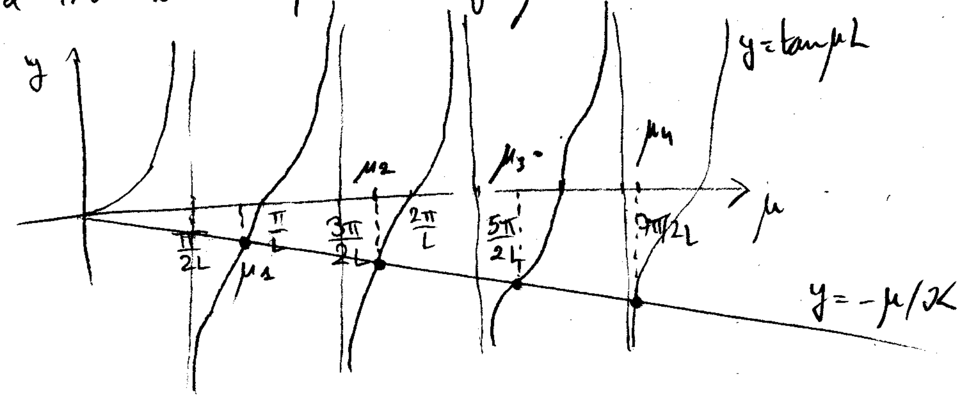
$\Rightarrow a = 0$. (trivial sol)

$k = -\mu^2$: $X(x) = a\cos\mu x + b\sin\mu x$

(BC1) $\Rightarrow$ $a = 0$

(BC2) $\Rightarrow$ $\mu\cos\mu L = -\mathcal{K}\sin\mu L \iff \tan\mu L = -\frac{\mu}{\mathcal{K}}$

gives a non linear eq to solve for μ:



can solve (numerically) for roots μ_n, $n \geq 1$

$\rightsquigarrow$ get solutions: $X_n(x) = \sin\mu_n x$

$T_n(t) = c_n \exp\left[-c^2\mu_n^2 t\right]$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left[-c^2\mu_n^2 t\right]\sin\mu_n x}$$ solves (2) by constructio

what about I.C?

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\mu_n x$$ $\sim$ Sine series but not quite "generalized Fourier series"

It turns out (Sturm-Liouville theory, §6) that

$\{\sin\mu_n x\}_{n=1}^{\infty}$ form an orthogonal system of functions

w.r.t inner product: $(u,v) = \int_0^L u(x)\,v(x)\,dx$.

thus

$$(f, \sin\mu_m x) = \sum_{n=1}^{\infty} c_n(\sin\mu_n x, \sin\mu_m x)$$

$$= c_m(\sin\mu_m x, \sin\mu_m x)$$

$$\Rightarrow c_m = \frac{(f, \sin\mu_m x)}{(\sin\mu_m x, \sin\mu_m x)}$$