§3.4 D'Alembert's Method

Recall 1DWEQ
$$\begin{cases} u_{tt} = c^2 u_{xx} & , \; 0<x<L, \; t>0 \quad \text{D.E.} \\ u(0,t) = u(L,t) = 0 & , \; t>0 \quad \text{B.C} \\ u(x,0) = f(x) & , \; 0<x<L \\ u_t(x,0) = g(x) & , \; 0<x<L \end{cases} \Bigg\} \text{ I.C.}$$

D'Alembert's solution:

$$\boxed{u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\, ds} \quad (*)$$

where f^* is 2L-periodic odd ext. of f, drop $*$ for notation simplicity below.
g^* " " " " " " g

consistency check: Can $u_n(x,t) = \sin\frac{n\pi}{L}x \cos\frac{cn\pi}{L}t$ be written as $(*)$?

$$\left.\begin{array}{l} \sin(a+b) = \sin a\cos b + \cos a \sin b \\ \sin(a-b) = \sin a \cos b - \cos a \sin b \end{array}\right\} \Rightarrow \sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

$$\Rightarrow u_n(x,t) = \frac{1}{2} \sin\frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)$$

Does $(*)$ solve 1DWEQ?

$$u_t = \frac{c}{2}\left[-f'(x-ct) + f'(x+ct)\right] + \frac{1}{2}\left[g(x+ct) + g(x-ct)\right]$$

since
$$\frac{\partial}{\partial t}\left[\int_{a(t)}^{b(t)} f(s;t)\,ds\right] = b'(t)\, f(b(t),t) - a'(t)\, f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(s;t)\, ds$$

$$u_{tt} = \frac{c^2}{2}\left[f''(x-ct) + f''(x+ct)\right] + \frac{c}{2}\left[g'(x+ct) - g'(x-ct)\right]$$

$$u_x = \frac{1}{2}\left[f'(x-ct) + f'(x+ct)\right] + \frac{1}{2}\left[g(x+ct) - g(x-ct)\right]$$

$$u_{xx} = \frac{1}{2}\left[f''(x-ct) + f''(x+ct)\right] + \frac{1}{2}\left[g'(x+ct) - g(x-ct)\right]$$

$$\Rightarrow u_{tt} = c^2 u_{xx}$$

what about B.C.?

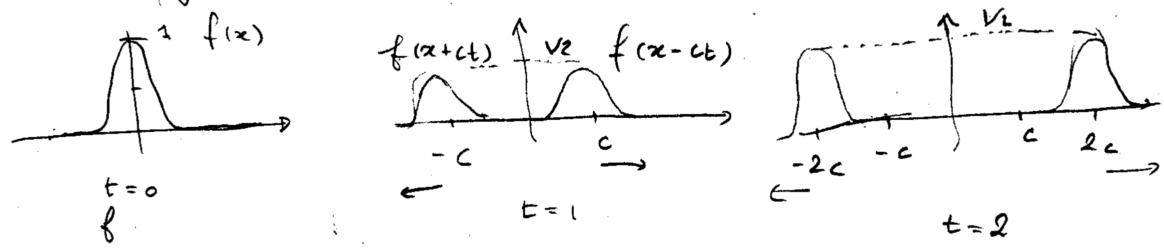
$$u(0,t) = \tfrac{1}{2}\left[f^*(-ct) + f^*(ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} g^*(s)\,ds = 0$$

(odd ext) (odd integrand)

etc... see homework. (3.4.14)

Physical interpretation

Assume $g(x)=0$. $\Rightarrow u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]$



humps propagate at speed c.

Let $G(x) = \int_a^x g^*(z)\,dz$ g is 2L-per

$$G(x+2L) - G(x) = \int_x^{x+2L} g^*(z)\,dz = \int_{-L}^{L} g^*(z)\,dz = 0$$

$\Rightarrow G$ is $2L$-periodic.

$$\Rightarrow u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

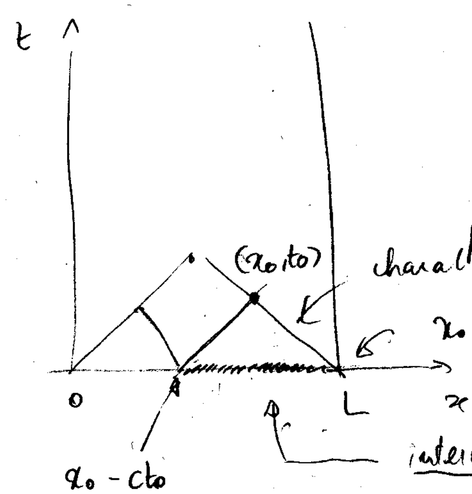
$$= \underbrace{\frac{1}{2}\left[f^*(x-ct) - \frac{1}{c}G(x-ct)\right]}_{\text{right propa term}} + \underbrace{\frac{1}{2}\left[f^*(x+ct) + \frac{1}{c}G(x+ct)\right]}_{\text{left propa term}}$$

Characteristic lines

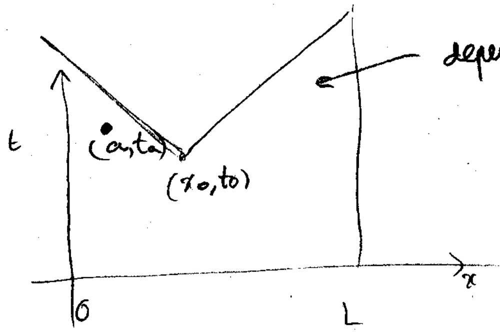
characteristics = curves where sol. is constant

$x - ct = x_0 - ct_0$

$x + ct = x_0 + ct_0$



interval of dependence: value of $u(x_0,t_0)$ can only be influenced by sol at this interval.

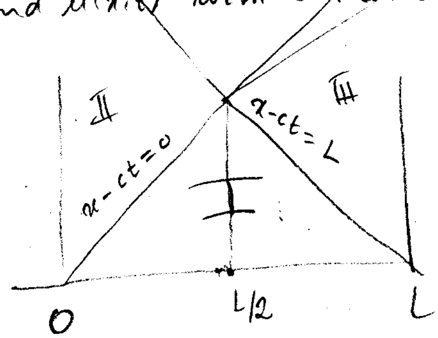


dependence cone (lightcone)

because waves (light) propagate at a finite speed, only observers in cone in xt plane can see disturbance inside (x_0, t_0)

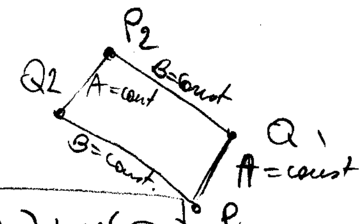
observer at $(a\ t_a)$ will not see disturbance!

If we do not want to use periodic ext of f and g, we can only find $u(x,t)$ with D'Alembert's method in a region I :



- It is possible to find values in regions II & III however this is complicated and hardly ever used because it works on 1D only.

Idea : prove relation :



(//) $\boxed{u(P_1) + u(P_2) = u(Q_1) + u(Q_2)}$

proof: $A(x,t) = \frac{1}{2}\left[f^*(x-ct) - \frac{1}{c} G(x-ct)\right]$ = RIGHT propa.

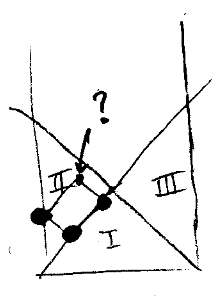
$B(x,t) = \frac{1}{2}\left[f^*(x+ct) + \frac{1}{c} G(x+ct)\right]$ = LEFT propa

$\Rightarrow u(x,t) = A(x,t) + B(x,t)$

$\Rightarrow$ $A(P_1) = A(Q_1)$; $A(P_2) = A(Q_2)$; $B(P_1) = B(Q_2)$; $B(P_2) = B(Q_1)$;

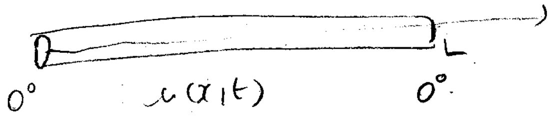
$u(P_1) + u(P_2) = A(P_1) + B(P_1) + A(P_2) + B(P_2)$
$= A(Q_1) + B(Q_2) + A(Q_2) + B(Q_1)$
$= u(Q_1) + u(Q_2)$

And identify such parallelograms.



all ● are known
can find ? using (//) identity.

§ 3.5 The One Dim. Heat eq (1DHEQ)



$u(x,t)$ = temperature distribution on a rod of length L, with ice bath on both ends.

$u(x,t)$ satisfies Heat Equation:

$$\begin{cases} u_t = c^2 u_{xx} & \\ u(0,t) = u(L,t) = 0^\circ & \text{B.C.} = \text{ice bath} \\ u(x,0) = f(x) & \text{I.C.} = \text{initial temperature distrib.} \end{cases}$$

Use method of separation of variable (it helps to see what we did last time in § 3.3)

Ansatz $u(x,t) = X(x)T(t)$

$$XT' = c^2 X''T \Rightarrow \underbrace{\frac{T'}{c^2T}}_{F(t)} = \underbrace{\frac{X''}{X}}_{G(x)} \underset{\text{must}}{=} k = \text{const indep of } x \text{ and } t.$$

We get 2 equations:

$$\begin{cases} X'' - kX = 0 \\ X(0) = X(L) = 0 \leftarrow \text{from B.C.} \end{cases}$$

$$k = -\mu^2, \quad \mu_n = \frac{n\pi}{L}$$

$$\boxed{X_n(x) = \sin \frac{n\pi}{L} x} \quad n = 1, 2, \dots$$

and $T' - kc^2T = 0$

$$T_n' + \left(\frac{cn\pi}{L}\right)^2 T_n = 0$$

$$\boxed{T_n(t) = b_n \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]}$$

$n = 1, 2, \dots$

Thus we get a fundamental mode:

$$u_n(x,t) = b_n \sin\left[\frac{n\pi}{L} x\right] \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right) \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$$

1DHEQ is homogeneous & lin.

Now what about initial conditions?

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \boxed{= \text{Sine Series of } f}$$

$$\Rightarrow b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

exp. decay of temp

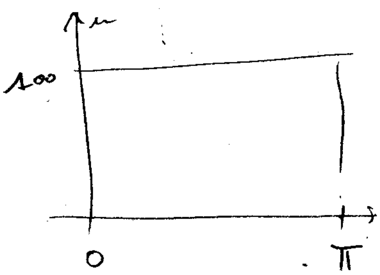
Summary

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

is solved by $u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$.

Example



$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 100 \end{cases}$$

$$\rightsquigarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \exp[-n^2 t]$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} 100 \sin nx \, dx$

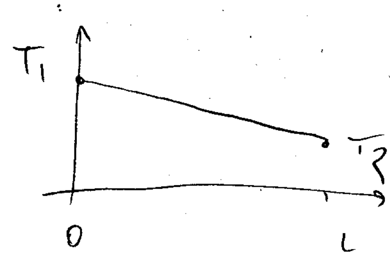
$$= -\frac{200}{\pi n} \cos nx \Big|_{x=0}^{\pi} = \frac{200}{n\pi}(1-(-1)^n)$$

$$u(x,t) = \sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t}}{2k+1} \sin (2k+1)x$$

$\boxed{\text{Show matlab code}}$

Steady state temp distrib:

$$u_t = 0 \Rightarrow u_{xx} = 0$$
$$\Rightarrow u(x) = Ax + B$$



$$\boxed{u(x)} = T_1 \frac{L-x}{L} + T_2 \frac{x-0}{L}$$

$$= \boxed{\frac{T_2 - T_1}{L} x + T_1}$$

Other Boundary Conditions:

$$(A)\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

① find steady state

$$s(x) = \frac{T_2 - T_1}{L} x + T_1$$

② shift solution by steady state

let $\boxed{v = u - s}$ then

$$v_t = u_t$$

$$v_{xx} = u_{xx}$$

$\Rightarrow$ v solves:

$$(B)\begin{cases} v_t = c^2 v_{xx} \\ v(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases}$$

And we know how to solve (B):

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin \frac{n\pi}{L} x$$

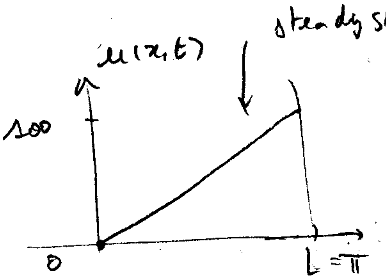
where $b_n = \frac{2}{L}\int_0^L (f(x) - s(x)) \sin \frac{n\pi}{L} x \, dx$

③ go back to original problem

$$\boxed{u(x,t) = v(x,t) + s(x)}$$

steady state temp $s(x) = \frac{100}{\pi} x$

Example:



$\Rightarrow$ I.C. for (B) is $100 - \frac{100}{\pi} x$

solve (B) for:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \left(100 - \frac{100}{\pi} x\right) \sin nx \, dx$$

$$= \frac{200}{n\pi}$$

$$\Rightarrow \boxed{u(x,t) = s(x) + \sum_{n=1}^{\infty} b_n \sin nx \; e^{-n^2 t}}$$

$$= \frac{100}{\pi} x + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} e^{-n^2 t}$$