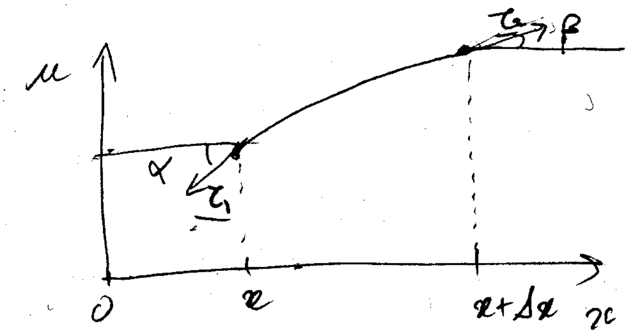
§ 3.2 Derivation of 1D Wave equation



Idea: Newton's second law $\sum F = ma$ on a piece of taut string.

- we have constant tension inside string ($|\tau_1| = |\tau_2|$) because length of string changes in a negligeable way with motion.
- we only need to look at vertical component as it can be argued that the horizontal movement is negligible

ρ = mass density kg/m of string (assumed constant).

mass × acceleration:

mass of segment $[x, x+\Delta x]$ × acc. of segment $[x, x+\Delta x]$

$$= \rho \Delta x \frac{\partial^2 u}{\partial t^2}(x,t)$$

sum of forces:

$$\tau \sin\beta - \tau \sin\alpha$$

Second law of motion:

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \tau(\sin\beta - \sin\alpha)$$
$$\approx \tau(\tan\beta - \tan\alpha)$$
$$= \tau\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right)$$

~~thus~~ as $\Delta x \to 0$:

$$\Rightarrow \frac{\rho}{\tau}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{\tau}{\rho}}$$

why? because of def of derivative

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \lim_{\Delta x \to 0} \frac{\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x}$$

note: since we are more interested in math here, you don't have to know this derivation.

§3.3 Solution to 1D WEQ using sep of var

1D WEQ:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & 0<x<L, \quad t>0 \\ u(0,t)=0;\ u(L,t)=0 & t>0 \quad \text{boundary conditions} \\ u(x,0)=f(x) & 0<x<L \quad \text{init pos of string} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & 0<x<L \quad \text{init vel of string} \end{cases} \Bigg\} \text{ init conditions}$$

(!) If you understand the principle behind this you understand the whole book.

We use the method of <u>separation of variables</u>.

<u>Idea</u>: • Seek solutions of the form

$$u(x,t) = X(x)\,T(t)$$

- plug ansatz into WEQ
- interpret and solve system of ODEs.
- solve original problem

Since $\dfrac{\partial^2 u}{\partial x^2} = X''T$ and $\dfrac{\partial^2 u}{\partial t^2} = XT''$ the WEQ becomes:

$$XT'' = c^2 X''T \iff \underbrace{\frac{T''}{c^2 T}}_{f(t)} = \underbrace{\frac{X''}{X}}_{g(x)} \quad \forall\, x,t$$

the only way we can have $f(t)=g(x)\ \forall\, x,t$ is if $f(t) = g(x) = k$ const

Thus we get 2 ODEs:

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{T''}{c^2 T} = k$$

i.e.

$$\begin{cases} X'' = kX & (1) \\ T'' = c^2 T & (2) \end{cases}$$

+ Boundary conditions given by B.C. of WEQ.

▶ Start with (1):

$$\begin{cases} X'' = kX \\ X(0)=X(L)=0 \end{cases}$$

B.C. $u(0,t) = X(0)T(t) = 0 \Leftrightarrow X(0)=0$
$u(L,t) = X(L)T(t) = 0 \Leftrightarrow X(L) = 0$

solution of this ODE depends on sign of k: (see Appendix A.2 for a refresh on this)

- case $k = \mu^2 > 0$

$$\left.\begin{aligned} X(x) &= a\cosh\mu x + b\sinh\mu x \\ X(0) &= a = 0 \\ X(L) &= b\underbrace{\sinh\mu L}_{\neq 0} = 0 \Rightarrow b = 0. \end{aligned}\right\} X(x) = 0$$

(trivial solution so this is not interesting case).

- Case $k = 0$

$$\left.\begin{aligned} X(x) &= ax + b \\ X(0) &= b = 0 \\ X(L) &= aL = 0 \Rightarrow a = 0. \end{aligned}\right\}$$

again trivial solution $X(x) = 0$

- case $k = -\mu^2 < 0$:

$X(x) = a\cos\mu x + b\sin\mu x$

$X(0) = a = 0$

$X(L) = b\sin\mu L = 0 \Rightarrow \mu L = n\pi,\ n \in \mathbb{Z},\ n \neq 0.$

$\Rightarrow \boxed{\mu_n = \frac{n\pi}{L}}$ however can restrict to $n \geq 1$ since $X_n = -X_{-n}$.

$\boxed{X_n(x) = \sin\frac{n\pi}{L}x}$ note: we take $b=1$ to make everything easy.

▶ Continue with (2):

$$T_n'' = -\left(c\frac{n\pi}{L}\right)^2 T_n \Rightarrow T_n(t) = b_n\cos\overbrace{\frac{cn\pi}{L}}^{\lambda_n \text{ in book}}t + b_n^*\sin\frac{cn\pi}{L}t.$$

Finally we get by construction that:

$$\boxed{u_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi}{L}x\left(b_n\cos\frac{cn\pi}{L}t + b_n^*\sin\frac{cn\pi}{L}t\right)}$$

solves WEQ and $u_n(0,t) = u_n(L,t) = 0$.

$u_n(x,t)$ = <u>normal modes</u> of wave equation

WEQ = linear & homogeneous we can use superposition principle

Supperposition principle: Any linear combination of $u_n(x,t)$ satisfies 1D WEQ + cdt° at end points.

To satisfy initial cdt° we look then for

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \left(b_n \cos\frac{n\pi}{L}ct + b_n^* \sin\frac{n\pi c}{L}t \right)$$

* Init condt° $u(x,0) = f(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x = f(x)$$

Thus b_n are coeff of 2L-per ODD expansion of $f(x)$ (Sine Series):

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\, dx} \quad , n \geq 1.$$

* Init condt° $u_t(x,0) = g(x)$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi}{L}x\right) \frac{n\pi c}{L}\left(-b_n \sin\frac{n\pi}{L}ct + b_n^* \cos\frac{n\pi}{L}ct\right)$$

$$\Rightarrow \quad u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n^* \sin\frac{n\pi}{L}x$$

$$\Rightarrow \quad \frac{n\pi c}{L} b_n^* = \text{Sine series coeff of } g(x)$$

$$= \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi}{L}x\, dx$$

$$\Rightarrow \quad \boxed{b_n^* = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi}{L}x\, dx} \quad , n \geq 1.$$

Summary

The sol to 1D WEQ:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & 0<x<L, \quad t>0 \\ u(0,t) = u(L,t) = 0, & t>0 \\ u(x,0) = f(x) & 0<x<L \\ u_t(x,0) = g(x) & 0<x<L \end{cases}$$

keep in mind: phases are dimensionless.

is:

$$u(x,t) = \sum_{n=1}^{\infty} \sin \underbrace{\frac{n\pi}{L}}_{1/\text{length}} x \left[b_n \cos \underbrace{\frac{n\pi}{L}}_{1/\text{length}} \overbrace{c}^{} t + b_n^* \sin \frac{n\pi}{L} c t \right]$$

1/length, 1/length, 1/s

where $b_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$

and $b_n^* = \dfrac{2}{cn\pi} \displaystyle\int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$

Example (see books example, here we do essentially problem 3.3.5

1DWEQ with $c=4$

$$f(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ (1-x) & \frac{1}{2} \le x \le 1 \end{cases} \qquad (L=1)$$

$$g(x) = 0 \implies b_n^* = 0$$

We need to compute Sine series of $f(x)$.

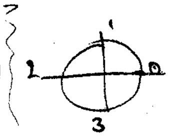
$$b_n = 2\int_0^1 f(x) \sin n\pi x \, dx = 2\int_0^{1/2} x \sin n\pi x \, dx + 2\int_{1/2}^1 (1-x) \sin n\pi x \, dx$$

$$= \cancel{\frac{-2}{n\pi} x \cos n\pi x \Big|_0^{1/2}} + \frac{2}{n\pi}\int_0^{1/2} \cos n\pi x \, dx \; \cancel{- \frac{2}{n\pi}(1-x)\cos n\pi x \Big|_{1/2}^1} - \frac{2}{n\pi}\int_{1/2}^1 \cos n\pi x \, dx$$

$$= \frac{2}{(n\pi)^2}\left[\sin n\pi x \Big|_0^{1/2} - \sin n\pi x \Big|_{1/2}^1 \right] = \boxed{\frac{4}{(n\pi)^2} \sin \frac{n\pi}{2}}$$

$n=$	$\sin n\pi/2$
0	0
1	1
2	0
3	-1

$$\implies u(x,t) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x) \cos(4(2k+1)\pi t)$$



show Matlab implementation.