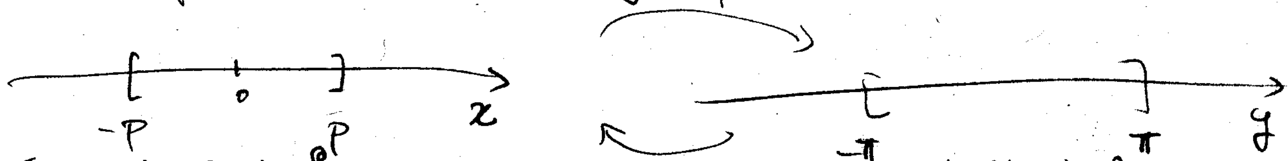
§2.3 Fourier series of functions with arbitrary periods

change of variables $\quad y = \frac{\pi}{p} x$



Inner prod: $(u,v) = \int_{-p}^{p} u\,v$ $\qquad$ Inner prod: $(u,v) = \int_{-\pi}^{\pi} u(y)\,v(y)\,dy$

2p-per fun $\qquad$ 2π-per fun.

$f(x) \qquad\qquad f\left(\frac{p}{\pi} y\right)$

$g\left(\frac{\pi}{p} x\right) \qquad\qquad g(y)$

$\left.\begin{matrix} 1, \cos\frac{\pi}{p}x, \dots, \cos n\frac{\pi}{p}x \\ \sin\frac{\pi}{p}x, \dots, \sin\frac{n\pi}{p}x \end{matrix}\right\}$ = orthogonal family $\qquad$ $\left.\begin{matrix} 1, \cos y, \dots, \cos ny \\ \sin y, \dots, \sin ny \end{matrix}\right\}$ = orthogonal family

proof:

$(1,1) = 2\pi$

$\left(\cos\frac{n\pi}{p}x, \cos\frac{m\pi}{p}x\right) = \int_{-p}^{p} \cos\frac{n\pi}{p}x \sin\frac{m\pi}{p}x \, dx$

cov $x = \frac{p}{\pi} y$

$= \frac{p}{\pi}\int_{-\pi}^{\pi} \cos ny \sin my \, dx$

$= \frac{p}{\pi}\begin{cases} \pi & \text{if } n=m \\ 0 & \text{otherwise} \end{cases} = \begin{cases} p & \text{if } n=m \\ 0 & \text{otherwise} \end{cases}$

$(1,1) = 2\pi$

$(\cos ny, \cos my) = \begin{cases} \pi & \text{if } n=m \\ 0 & \end{cases}$

$(\sin ny, \sin my) = \begin{cases} \pi & \text{if } n=m \\ 0 & \end{cases}$

$(\sin ny, \cos my) = 0$

Similarly: $\left(\sin\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right) = \begin{cases} p & \text{if } n=m \\ 0 & \text{otherwise} \end{cases}$

$\left(\sin\frac{n\pi}{p}x, \cos\frac{m\pi}{p}x\right) = 0$

Theorem 1: Let f be a 2p-periodic piece smooth function. Its Fourier series expansion is:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x$$

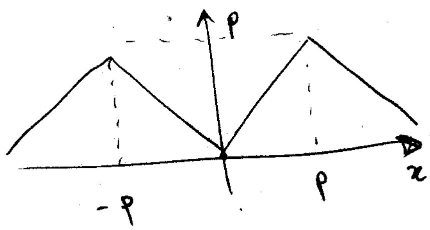
where: $\quad a_0 = \frac{(f,1)}{(1,1)} = \frac{1}{2p}\int_{-p}^{p} f(x)\,dx$

$$a_n = \frac{\left(f, \cos\frac{n\pi}{p}x\right)}{\left(\cos\frac{n\pi}{p}x, \cos\frac{n\pi}{p}x\right)} = \frac{1}{p}\int_{-p}^{p} f(x) \cos\frac{n\pi}{p}x \, dx$$

and $b_n = \dfrac{(f, \sin \frac{n\pi}{p} x)}{(\sin \frac{n\pi}{p}x, \sin\frac{n\pi}{p}x)} = \dfrac{1}{p}\displaystyle\int_{-p}^{p} f(x)$

The series converges pointwise to $\dfrac{f(x+)+f(x-)}{2}$.

Example.

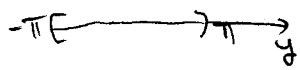


$f(x) = |x|$ if $-p \le x \le p$, $2p$-periodic

method 1: compute integrals for a_i, b_i.
(See Example 2.3.1)

method 2: use Fourier series for the 2π-per function:

$x = \frac{p}{\pi} y$



$g(y) = |y|$ if $-\pi \le y \le \pi$:

$$g(y) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right) \cos n y$$

By stretching and rescaling g:

$$g\left(\frac{\pi}{p}x\right) = \frac{\pi}{p}|x| \text{ if } -\pi \le \frac{\pi}{p}x \le \pi$$
$$= \frac{\pi}{p}|x| \text{ if } -p \le x \le p$$

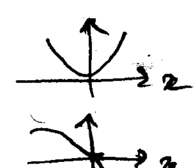
thus $f(x) = \dfrac{p}{\pi}\, g\left(\dfrac{\pi}{p}x\right) = \dfrac{p}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2p}{\pi^2 n^2}\left((-1)^n - 1\right) \cos \dfrac{n\pi}{p} x$

Even and odd functions

Recall:

f is even $\Leftrightarrow$ $f(-x) = f(x)$ $\forall x$

f is odd $\Leftrightarrow$ $f(-x) = -f(x)$ $\forall x$



$$\int_{-p}^{p} f(x)\,dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_0^p & \text{if } f \text{ is even} \end{cases} \qquad (\text{cov})$$

Also:
$$\begin{cases} \text{even} \times \text{even} = \text{even} \\ \text{even} \times \text{odd} = \text{odd} \\ \text{odd} \times \text{odd} = \text{even} \end{cases}$$

Implications for Fourier Series

- If f is a $2p$-per **even** function: $b_n = \frac{1}{p}\int_{-p}^{p} \overbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin \frac{n\pi}{p} x}_{\text{odd}}}^{\text{odd}}\, dx = 0$

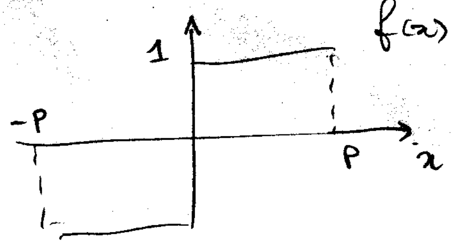
$$\Leftrightarrow \underset{\text{even}}{f(x)} = \underset{\text{even}}{a_0} + \sum_{n=1}^{\infty} a_n \underset{\text{even}}{\cos} \frac{n\pi}{p} x.$$

- If f is a $2p$-per **odd** function: $a_0 = \frac{1}{2p}\int_{-p}^{p} \underbrace{f(x)}_{\text{odd}}\, dx = 0$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos \frac{n\pi}{p} x}_{\text{even}}}_{\text{odd}}\, dx = 0$$

$$\Leftrightarrow \underset{\text{odd}}{f(x)} = \sum_{n=1}^{\infty} b_n \underset{\text{odd}}{\sin} \frac{n\pi}{p} x.$$

cf Examples 2.3.4, 2.3.5 and problem 2.3.1



f = odd function $\Rightarrow a_n = 0$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx$$

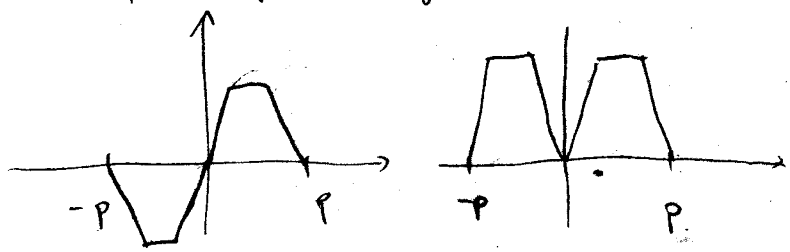
$$= \frac{2}{p}\int_{0}^{p} \sin \frac{n\pi}{p} x\, dx = -\frac{2}{p}\frac{p}{n\pi} \cos \frac{n\pi}{p}$$

$$= -\frac{2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$\boxed{\Rightarrow f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin \frac{(2k+1)\pi x}{p}}$$

Half range expansions: the cosine and sine series

Idea: A function f defined on $[0,p]$ can be extended to a $2p$ periodic function as follows:



odd extension — even extension

Cosine series = Fourier series of even extension

Sine series = F.S. of odd extension.

Theorem (Half range expansions) Let $f(x)$ be a pcws smooth func def on $[0,p]$. The cosine series expansion of $f(x)$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x \quad \text{for } 0 \leq x \leq p$$

(EVEN by construction)

$$\text{where } a_n = \frac{\left(f, \cos\frac{n\pi x}{p}\right)}{\left(\cos\frac{n\pi}{p}x, \cos\frac{n\pi}{p}x\right)} = \frac{1}{\underbrace{\int_{-p}^{p}\left(\cos\frac{n\pi}{p}x\right)^2 dx}} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p} dx$$

same formula as Fourier series, but we assume that on $[-p,0]$ f has been extended as an EVEN function: $f(-x) = f(x)$

$$\Rightarrow \boxed{\begin{aligned} a_0 &= \frac{1}{\cancel{2}p} \cancel{2} \int_0^p f(x)\, dx \\ a_n &= \frac{1}{p}\, 2 \int_0^p f(x) \cos \frac{n\pi x}{p} dx \end{aligned}}$$

Similarly, the sine series expansion of $f(x)$ is:

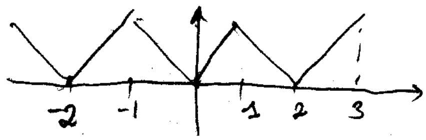
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p} \quad \text{for } 0 \leq x \leq p$$

(ODD by construction)

$$\text{where } \boxed{b_n = \frac{\left(f, \sin\frac{n\pi x}{p}\right)}{\left(\sin\frac{n\pi x}{p}, \sin\frac{n\pi x}{p}\right)} = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi x}{p} dx}$$

Example: Consider $f(x) = x$ for $0 \le x \le 1$

even^{n} extension



$$a_0 = \frac{2}{2}\int_0^1 x\,dx = \frac{1}{2}$$

$$a_n = \frac{2}{1}\int_0^1 x \cos n\pi x\,dx$$

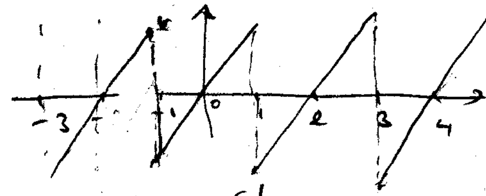
$$= 2x\,\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx$$

$$= +\frac{2}{(n\pi)^2}\cos n\pi x\Big|_0^1 = \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} -\dfrac{4}{(n\pi)^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$\boxed{x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos (2k+1)\pi x}$$

odd^{n} extension



$$b_n = \frac{2}{1}\int_0^1 x \sin n\pi x\,dx$$

$$= -2x\,\frac{\cos n\pi x}{n\pi}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos n\pi x\,dx$$

$$= -\frac{2(-1)^n}{n\pi} + \cancel{\frac{2}{(n\pi)^2}\sin n\pi x\Big|_0^1}$$

$$\Rightarrow \boxed{x = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin n\pi x}$$