§ 2.1 Periodic functions

Def. A function $f$ is periodic with period $T$ iff

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}.$$  
(aka T-periodic function)

Example:

$$\sin(x), \quad \cos(2x), \quad \frac{\pi}{2n}$$

Note:

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT).$$

$f$ is also an $nT$-periodic function for $n \geq 1$.

Note: A $T$-periodic function can be defined in many different but equivalent ways:

$$f(x) = |x| \quad \text{for} -1 \leq x \leq 1$$

$$f(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq 1 \\ 2-x & \text{for} \quad 1 < x \leq 2 \end{cases}$$

$T = 2$

Def. (right, left limit):

$$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c+h)$$

$$\lim_{x \to c^-} f(x) = \lim_{h \to 0} f(c-h)$$

Def. (continuity): Let $f: \mathbb{R} \to \mathbb{R}$ be a function. The following props are equivalent:

(a) $f$ is continuous at $c$

(b) $\lim_{x \to c} f(x) = c$

(c) $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$
Def (piecewise continuous)
A function $f$ is piecewise continuous on an interval $[a,b]$ if:
1) $f(a+)$ and $f(b-)$ exist
2) $f$ is defined and continuous on $(a,b)$ except at a finite number of points in $(a,b)$ where the left and right limits exist.

For periodic functions:
$f$ is periodic on a period $\Rightarrow$ $f$ is periodic on $\mathbb{R}$ (e.g., $[0, T]$)

However,

For a periodic function to be continuous on $\mathbb{R}$ we need:
1) $f$ is continuous on $[0, T]$
2) $f(0+) = f(T-)$

Def (piecewise smooth)
$f$ is piecewise smooth on $[a,b]$ iff $f$ and $f'$ are piecewise continuous on $[a,b]$.

\[ f \]

\[ f' \]

\[ \sqrt{12} \]

piecewise continuous but not piecewise smooth.
Theorem (Integral Mean Value)

Let $f$ be a $T$-periodic function, then

$$\int_0^T f(x) \, dx = \int_{a-T}^a f(x) \, dx \quad \forall a \in \mathbb{R}$$

**Proof:** assuming $f$ is continuous but holds in general,

$$F(a) = \int_a^{a+T} f(x) \, dx$$

$$F'(a) = f(a+T) - f(a) = 0$$

$$\Rightarrow \ F(a) = \text{constant}$$

**Example:** with $f(x) = |x|$, $-1 < x < 1$, 2-periodic function

Compute:

- $$\int_{-1}^1 -x \, dx = \frac{1}{2} \int_{0}^2 x \, dx = 1$$

(b) $$\int_{-3}^{2} f(x) \, dx = \int_{-3}^{-1} f(x) \, dx + \int_{-1}^{1} f(x) \, dx + \int_{1}^{3} f(x) \, dx = N$$

$L^2[a,b]$ inner product

Let $u, v$ be real-valued functions defined on $[a, b]$. The inner product of $u$ and $v$ is:

$$(u, v) = \int_a^b u(x) \overline{v(x)} \, dx$$

**Def (1 of functions):** Two functions $u, v$ are said to be orthogonal if $(u, v) = 0$ (think vectors $\perp$)

An important orthogonal family of functions is the trigonometric system defined on $[0, 1]$

$$1, \cos x, \cos 2x, \ldots, \cos nx, \ldots$$

$$\sin x, \sin 2x, \ldots, \sin nx, \ldots$$
Why is this called an orthogonal family? Since:

\[
\begin{align*}
\cos(mx, \cos nx) & = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n > 0 \\ -2\pi & \text{if } m = n = 0. \end{cases} \\
\sin(mx, \sin nx) & = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}
\end{align*}
\]

**Proof:** Trig identities. e.g.

\[
\begin{align*}
\cos(m+n)x &= \cos mx \cos nx - \sin mx \sin nx \\
\cos(m-n)x &= \cos mx \cos nx + \sin mx \sin nx
\end{align*}
\]

\[
\Rightarrow \quad \cos mx \cos nx = \frac{1}{2} \left[ \cos((m+n)x) + \cos((m-n)x) \right]
\]

\[
\Rightarrow \quad \cos(mx, \cos nx) = \frac{1}{2} \int_{-\pi}^{\pi} \cos(mx + nx) + \cos(mx - nx) \, dx
\]

\[
= \frac{1}{2(m+n)} \sin(m+n)x + \frac{1}{2(m-n)} \sin(m-n)x, \quad m \neq n
\]

and:

\[
\cos(nx, \cos nx) = \frac{1}{2} \int_{-\pi}^{\pi} 1 + \cos(2nx) \, dx = \frac{1}{2} 2\pi.
\]

\[
\text{Ok...}
\]

**Fourier Series (§2.2)**

**Idea:** the trig system \( \cos x \cos 2x \ldots \sin x \sin 2x \ldots \) is an orthogonal basis of functions. Any suitable function (in particular piecewise smooth functions) can be essentially expanded in this basis:

\[
M(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]

\[
a_i, b_i = \text{coeff of } M \text{ in this basis}
\]