§2.1 Periodic functions

Def A function f is periodic with period T iff

$$f(x+T) = f(x) \quad \forall\, x \in \mathbb{R}.$$

(aka T-periodic function)

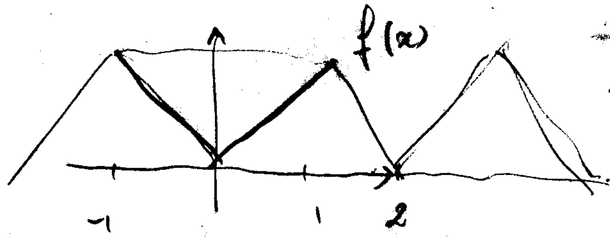
Example: $\sin(x)$ $\quad$ $\sin(nx)$

2π $\quad$ $\frac{2\pi}{n}$

Note: $f(x) = f(x+T) = f(x+2T) = \dots = f(x+nT)$

f is also an nT-periodic function for $n \geq 1$.

Note A T-periodic function can be defined in many different but equivalent ways



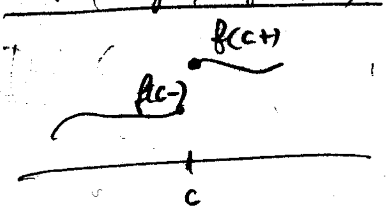
$T=2$

$f(x) = |x|$ for $-1 \leq x \leq 1$

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2-x & \text{for } 1 < x \leq 2 \end{cases}$$

(depends on which interval you are initially for "copy paste"

Def (right, left limit)



$$f(c+) = \lim_{x \to c+} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(c+h)$$

$$f(c-) = \lim_{x \to c-} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(c-h)$$

Def (continuity): Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The following prop are equivalent

(a) f is continuous at c

(b) $\lim_{x \to c} f(x) = c$

(c) $\lim_{x \to c+} f(x) = \lim_{x \to c^-} f(x) = f(c)$

Def (piecewise continuous)

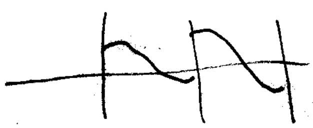
A function f is piecewise continuous on an interval $[a,b]$ if:

i) $f(a+)$ and $f(b-)$ exist

ii) f is defined and continuous on (a,b) except at a finite number of points in (a,b) where the left and right limits exist.

For periodic functions: f is pcws cont on a period (e.g. $[0,T]$) $\Leftrightarrow$ f is pcws cont on $\mathbb{R}$

however



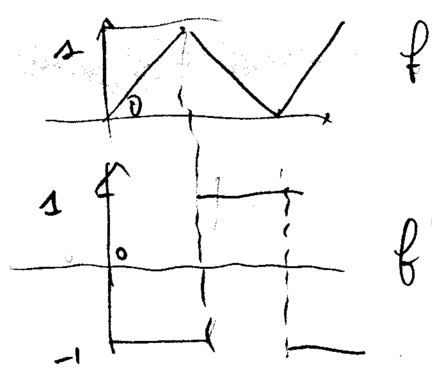
: f is cont on a period (e.g $[0,T]$) $\Rightarrow$ f is pcws cont on

For a periodic function to be cont. on $\mathbb{R}$ we need:

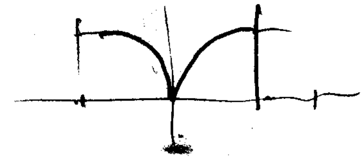
i) f is cont on $[0,T]$

ii) $f(0+) = f(T-)$

Def (piecewise smooth) f is pcws smooth on $[a,b]$ iff f and f' are pcws cont on $[a,b]$.



pcws smooth



$\sqrt{|x|}$

piecewise cont but not pcws smooth.

Theorem (integral over period)

Let f be a T-periodic function then

$$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx \quad \forall a \in \mathbb{R}$$

proof: assuming f is cont. (but holds in general)

$$F(a) = \int_a^{a+T} f(x)\,dx$$

$$F'(a) = f(a+T) - f(a) = 0$$

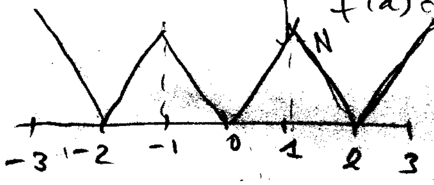
$$\Rightarrow F(a) = \text{const.}$$

Example: with $f(x) = |x|$, $-1 \le x < 1$, 2-periodic function

compute: (a) $\int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = 1$

(b) $$\int_{-N}^{N} f(x)\,dx = \overbrace{\int_{-N}^{-N+2} f(x)\,dx + \int_{-N+2}^{-N+4} f(x)\,dx + \cdots + \int_{N-2}^{N} f(x)\,dx}^{N \text{ times}}$$

$$= N.$$



$L^2[a,b]$ inner product Let u, v be real valued functions defined on $[a,b]$. The inner product of u and v is:

$$(u, v) = \int_a^b u(x)\,v(x)\,dx.$$

Def ($\perp$ of functions): Two functions u, v are said to be orthogonal if $(u, v) = 0$ (think vectors $v \perp u$)

An important orthogonal family of functions is the trigonometric system (defined on $[-\pi, \pi]$)

$$1 \quad \cos x, \cos 2x, \ldots, \cos mx, \ldots$$
$$\sin x, \sin 2x, \ldots, \sin mx, \ldots$$

Why is this called an orthogonal family? Since:

$$(\cos mx, \cos nx) = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n > 0 \\ 2\pi & \text{if } m = n = 0. \end{cases}$$

$$(\sin mx, \sin nx) = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$(\cos mx, \sin nx) = 0 \quad \forall n, m.$$

proof: trig identities. e.g.

$$\cos(m+n)x = \cos mx \cos nx - \sin mx \sin nx$$
$$\cos(m-n)x = \cos mx \cos nx + \sin mx \sin nx$$

$$\Rightarrow \quad \cos mx \cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$$

$$\Rightarrow \quad (\cos mx, \cos nx) = \frac{1}{2}\int_{-\pi}^{\pi} \cos m+nx + \cos(m-n)x \, dx$$
$$= \frac{1}{2(m+n)}\sin(m+n)x + \frac{1}{2(m-n)}\sin(m-n)x, \quad m \neq$$
$$= 0$$

and: $(\cos nx, \cos nx) = \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos(2nx)\, dx = \frac{1}{2} 2\pi.$

etc...

Fourier Series (§2.2)

Idea: the trig system $1 \quad \cos x \quad \cos 2x \; \cdots$
$ \quad \sin x \quad \sin 2x \; \cdots$

is an orthogonal basis of functions. Any suitable function (in particular pcws smooth functions) can be essentially expanded in this basis:

$$u(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

a_i, b_i = coeff of u in this basis