MATH3150 PDEs for Engineers

§1.1 - 1.2, 3.1

Examples of PDEs	model	order	lin	
$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$	advection eq	1	y	
$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	wave eq.	2	y	1D
$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	heat eq	2	y	
$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$	Burger eq. ~ Navier Stokes	1	n	n
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Δu	Laplace eq diffusion, acoustics.. flow in porous medium	2	y	
$\Delta u = f$	Poisson eq		y	

order of PDE : highest order of differentiation

linear equation: $L(u) = f$ where

L = linear diff. op i.e. satisfies:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \quad \text{for all } \alpha, \beta \in \mathbb{R}$$

u, v suitable fn.

homogeneous eq, all terms involve unknown.

e.g: $L(u) = f$ is linear, non-homog

Poisson eq is

Laplace homog

Example: advection eq : linear homog first order DE

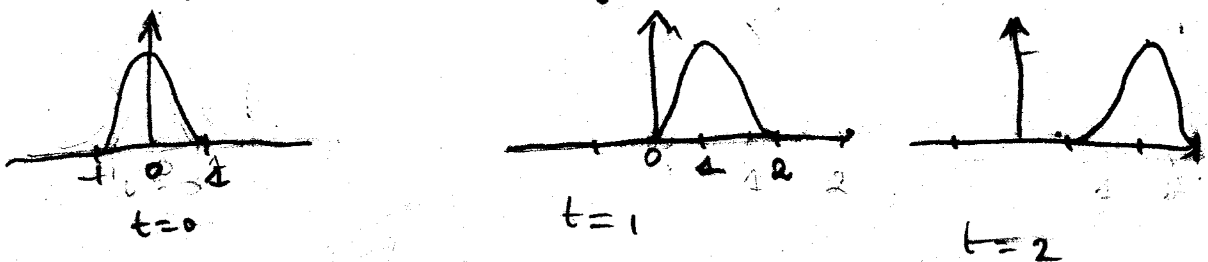
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

has sol of the form:

$$u(x,t) = f(x-t)$$

verifying: $\frac{\partial u}{\partial t} = -f'(x-t)$, $\frac{\partial u}{\partial x} = f'(x-t)$

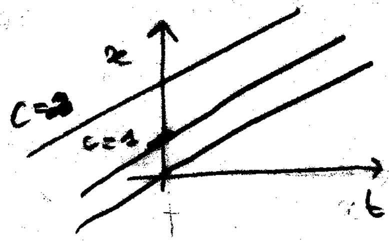
meaning: $u(x,0) = f(x)$



changes origin of initial cond to t

propagating waveform.

Also solution is constant on lines $x - t = c$



= "characteristics" or "characteristic curves".

method of characteristics (lw problem)

(*) $\frac{\partial u}{\partial x} + p(x,y)\frac{\partial u}{\partial y} = 0$, by finding "characteristic

a curves in the xy plane where $u(x,y)$ does not change

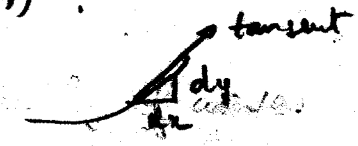
Rewrite (*) as: $\nabla u \cdot \begin{pmatrix} 1 \\ p(x,y) \end{pmatrix} = 0$

$\Leftrightarrow$ dir. der of u in dir $\begin{pmatrix} 1 \\ p(x,y) \end{pmatrix} = 0$

$\Leftrightarrow$ $u(x,y)$ does not change in dir. $\begin{pmatrix} 1 \\ p(x,y) \end{pmatrix}$

key: $\begin{pmatrix} 1 \\ p(x,y) \end{pmatrix}$ = tangent to characteristics

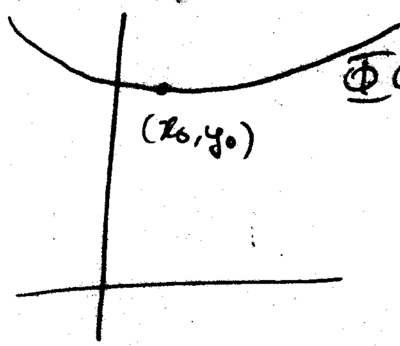
What are the curves with tangent $(1, p(x,y))$?



Solve

$$\frac{dy}{dx} = \frac{f(x,y)}{1} = \text{tangent slope}$$

to get a family of curves of the form $\Phi(x,y) = C$



$\Phi(x,y) = \Phi(x_0,y_0)$

val of u can be diff from char to ch

$$u(x,y) = f(C)$$
$$= f(\Phi(x,y))$$

example:

- adv eq: $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ (tx-plane)

characteristics: $\frac{dx}{dt} = 1 \Rightarrow x = t + c$

$\Rightarrow \Phi(t,x) = x - t =$

$\Rightarrow u(x,t) = f(x - t)$

- $\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

characteristics satisfy: $\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + C$

$\Phi(x,y) = y - \frac{x^2}{2} = C$

$\Rightarrow u(x,y) = f(C) = f(y - \frac{x^2}{2})$.

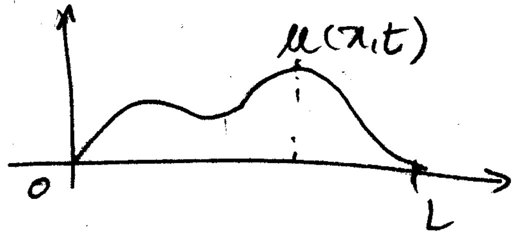
check:

$$\frac{\partial u}{\partial x} = -\cancel{2}x f'(y - x^2)$$

$$\frac{\partial u}{\partial y} = x f'(y - x^2)$$

$$u_x + x u_y = 0$$

Vibrating string



$u(x,t) =$ displacement away equil. of string at pos x and time

governed by 1D Wave eq

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

extremely important.
oil exploration

Linear 2nd order homog PDE

$L(u) = 0$ where $L(u) = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$

initial conditions:

$u(x,0) = f(x)$ (shape at $t=0$)

$\frac{\partial u}{\partial t}(x,0) = g(x)$ (vel of every piece of string at $t=0$)

boundary conditions (string fixed at end points)

$u(0,t) = 0$
$u(L,t) = 0$

Here is an important family of sol. to 1D WEQ:

$$u_n(x,t) = \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} \qquad n = 1,2,\ldots$$

check:

$$\frac{\partial^2 u_n}{\partial t^2} = -\left(\frac{n\pi c}{L}\right)^2 \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L}$$

$$\frac{\partial^2 u_n}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 \quad '' \quad ''$$

What about BC?

$$u_n(0,t) = \sin(0)\cos(\cdots) = 0$$
$$u_n(L,t) = \sin(n\pi)\cos(\cdots) = 0$$

what about IC?

$$u_n(x,0) = \sin\frac{n\pi}{L}x \, \cancel{\cos(0)}^{\,1}$$

$$\frac{\partial u_n}{\partial t}(x,0) = -\frac{cn\pi}{L}\sin\frac{n\pi}{L}x \quad \sin(0) = 0$$

zero initial velocity and very particular init pos: this family is important because principle of superposition + Fourier series shows that can be used to construct sol to WE with arbitrary init

Assume we have $u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$

then:
$$L(u) = L\left(\sum_{n=1}^{\infty} b_n u_n(x,t)\right) \quad \text{Linearity}$$
$$= \sum_{n=1}^{\infty} L(u_n(x,t))$$
$$= 0$$

Moreover:
$$u(0,t) = \sum_{n=1}^{\infty} b_n u_n(0,t) = 0$$

& similarly $u(L,t) = 0$

$\Rightarrow$ u solves also 1DWEQ with IC:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n u_n(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$
$$0 = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \frac{\partial u_n}{\partial t}(x,0) = \sum_{n=1}^{\infty} 0$$

Reverse engineering:

if an initial pos of string can be written in form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

(Fourier series)

then sol to:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

is given by:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi}{L} x$$

Goal in this class: find particular sol to certain DEs and use linearity (principle of superposition = fancier name) to construct general sol

For completeness:

<u>pple of superp:</u>

Let L be a linear diff op

and $L(u) = 0$ (1) be a linear homog diff eq

then if u, v are sol to (1)

$\Rightarrow$ $\alpha u + \beta v$ is a sol to (1) for any α, β

proof:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) = 0.$$