### Examples of PDEs

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**order of PDE**: highest order of differentiation

**linear equation**: \( L(u) = f \) where

\[ L = \text{linear diff. op i.e. satisfies:} \]

\[ L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \text{ for all } \alpha, \beta \in \mathbb{R} \]

**homogeneous eq.** all terms involve unknown.

- \( L(u) = f \) is linear, non-homog
- Poisson eq. is

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Example: advection eq - linear homog first order DE
\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \]

has sol of the form:
\[ u(x,t) = f(x-t) \]

Verifying:
\[ \frac{\partial u}{\partial t} = -f'(x-t), \quad \frac{\partial u}{\partial x} = f'(x-t) \]

meaning:
\[ u(x,0) = f(x) \]

changes origin of initial cond to \( t \)
propagating waveform.

Also solution is constant on lines \( x-t = c \)

\( c \) = "characteristics" or "characteristic curves."

Method of Characteristics (RW problem)

\[ \frac{\partial u}{\partial x} + p(x,y) \frac{\partial u}{\partial y} = 0 \]

by finding "characteristic" a curve on the \( xy \) plane where \( u(x,y) \) does not change

Rewrite (1) as:
\[ \nabla \cdot (\nabla u) = 0 \]

(2) div. div of \( u \) in \( x \) \( \frac{\partial u}{\partial x} = 0 \)
(3) \( u(x,y) \) does not change in \( y \) \( \frac{\partial u}{\partial y} = 0 \)

Key:
\[ \left( \frac{\partial}{\partial (p(x,y))} \right) = \text{tangent to characteristic} \]
Check:

\[ \frac{2x}{y} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{2}{2y} = \frac{2}{y} \]

Characteristic equation:

\[ \frac{dx}{dt} + \frac{2}{y} x = 0 \]

\[ \frac{dy}{dt} = y \]

Characteristic solution:

\[ y(t) = Ce^{2t} \]

\[ x(t) = C'e^{t} \]

\[ \phi(t, y) = \frac{y}{x} \]

To get a family of curves of the form \( \phi(x, y) = C \):

\[ \frac{dy}{dt} = \frac{y}{x} \]

Separate variables:

\[ \frac{dy}{y} = \frac{dx}{x} \]

Integrate:

\[ \ln y = \ln x + C \]

\[ y = Cx \]
Vibrating string

\[ u(x,t) \]

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Linear 2nd order homog PDE

\[ L(u) = 0 \quad \text{where} \quad L(u) = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \]

Initial conditions:

- \[ u(x,0) = f(x) \quad \text{(shape at } t=0) \]
- \[ \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{(vel of every piece of string at } t=0) \]

Boundary conditions (string fixed at end points):

- \[ u(0,t) = 0 \]
- \[ u(L,t) = 0 \]

Here is an important family of soln to 1D wave:

\[ u_n(x,t) = \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} \quad n = 1, 2, \ldots \]

check:

\[ \frac{\partial^2 u_n}{\partial t^2} = -\left(\frac{n\pi c}{L}\right)^2 \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} \]

\[ \frac{\partial^2 u_n}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} \]

extremely important.
What about $BC$?

\[ u_n(0, t) = \sin(0) \cos(\cdot) = 0 \]
\[ u_n(L, t) = \sin(n\pi) \cos(\cdot) = 0 \]

What about IC?

\[ u_n(x, 0) = \sin \left( \frac{n\pi x}{L} \right) \cos(0) \]
\[ \frac{\partial u_n}{\partial t}(x, 0) = \frac{n\pi}{L} \sin \frac{n\pi x}{L} \]

zero initial velocity and very particular initial position.

This family is important because principle of superposition + Fourier series threads that can be used to construct trial to W.E. with arbitrary initial

Assume we have $u(0, t) = \sum_{n=1}^{\infty} u_n(x, t)$

then:

\[ \mathcal{L}(u) = \int_0^L \left( \sum_{n=1}^{\infty} \mathcal{L}(u_n(x, t)) \right) \text{d}x \]

\[ = \sum_{n=1}^{\infty} \mathcal{L}(u_n(x, t)) \]

Moreover:

\[ u(0, t) = \sum_{n=1}^{\infty} u_n(0, t) = 0 \]

\[ u(L, t) = \sum_{n=1}^{\infty} u_n(L, t) = 0 \]

\[ \Rightarrow u \text{ solves also IDWEQ with IC.} \]

\[ f(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n u_n(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

\[ 0 = \frac{\partial u(x, 0)}{\partial x} = \sum_{n=1}^{\infty} b_n \frac{\partial u_n(x, 0)}{\partial x} = \sum_{n=1}^{\infty} b_n 0 \]
Revers engineering:

If an initial condition of string can be written in the form

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

then solve to:

\[
\begin{cases} 
\mu_{tt} = c^2 \mu_{xx} \\
\mu(0,t) = \mu(L,t) = 0 \\
\mu(x,0) = f(x) \\
\mu_t(x,0) = 0 
\end{cases}
\]

is given by:

\[ \mu(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \]

Goal in this class: find particular sol to certain DEs and use linearity (principle of superposition = revers, more) to construct a general sol.

For completeness:

*Rule of superposition*

Let \( L \) be a linear diff op and \( aL(u) = 0 \) be a linear homog diff eq.

Then if \( u, v \) are sol to \( (a) \)

\[ u + \beta v \] is sol to \( (1) \) for any \( \beta \).

**Proof:**

\[ L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) = 0. \]