Total: 150/150 points

Problem 1 (25 pts) Consider the 1D heat equation with homogeneous Dirichlet boundary conditions,
\[
\begin{cases}
u_t = 4u_{xx}, & 0 < x < 1 \text{ and } t > 0, \\
u(0, t) = u(1, t) = 0, & t > 0, \\
u(x, 0) = f(x), & 0 < x < 1.
\end{cases}
\]  
(1)

(a) Use separation of variables with \( u(x, t) = X(x)T(t) \) to show that a general solution to (1) is
\[
u(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \exp[-(2n\pi)^2 t].
\]
Specify what are the coefficients \( b_n \) in terms of \( f(x) \).

(b) Solve (1) with
\[
f(x) = \begin{cases} 
100 & \text{if } 0 < x \leq \frac{1}{2}, \\
0 & \text{if } \frac{1}{2} < x \leq 1.
\end{cases}
\]

(c) Find the steady state solution \( s(x) \) to
\[
\begin{cases}
u_t = 4\nu_{xx}, & 0 < x < 1 \text{ and } t > 0, \\
u(0, t) = 0, \nu(1, t) = 100, & t > 0, \\
u(x, 0) = g(x), & 0 < x < 1.
\end{cases}
\]  
(2)

(d) Solve (2) with \( g(x) = f(x) + s(x) \) by using your answer to (b).

(e) Plugging \( u(x, t) = X(x)T(t) \) into PDE:
\[
XT' = 4X''T 
\Rightarrow \frac{X''}{X} = \frac{T'}{4T} = \text{constant} < 0 \text{ to get decay}
\]
\[
\Rightarrow X'' + \lambda^2 X = 0
\]
Solving for \( X \) we get,
\[
X(x) = a \cos \lambda x + b \sin \lambda x
\]
\[
x(0) = a = 0 \Rightarrow a = 0
\]
\[
x(1) = b \sin \lambda = 0 \Rightarrow b = 0 \text{ (trivial sol)}
\]
\[
\Rightarrow \lambda = \lambda_n = n\pi
\]
\[
\Rightarrow X_n(x) = \sin n\pi x
\]
The problem for $T$ is:

$$T_n' = -4\frac{(\pi n)^2}{\nu^2} T_n \Rightarrow T_n(t) = b_n \exp\left[-\frac{(2\pi n)^2}{\nu^2} t\right]$$

Let $X_n(x)$ be the $n$th eigenfunction of $T_n(x)$, it satisfies (1) by construction and so does its sum by the superposition principle:

$$(\mu(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \exp\left[-\frac{(2\pi n)^2}{\nu^2} t\right]$$

Using Initial Conditions, as

$$\mu(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

The $b_n$ are time series coefficients of $f(x)$:

$$(f, \sin(n\pi x)) = b_n \left(\int_0^1 \sin(n\pi x) \cos(n\pi x) \, dx\right)$$

Besides, $(\sin(n\pi x), \sin(m\pi x)) = \frac{1}{n\pi}\int_0^1 \sin(n\pi x) \sin(m\pi x) \, dx = \frac{1}{2}$.

Thus

$$b_n = 2\int_0^1 f(x) \sin(n\pi x) \, dx$$

(6) With the given $f(x)$:

$$b_n = 2\cos\frac{n\pi}{2} \int_0^1 \sin(n\pi x) \, dx = 2\cos\frac{n\pi}{2} \left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1$$

$$b_n = \frac{2\cos\frac{n\pi}{2}}{n\pi} \left(1 - \cos\frac{n\pi}{2}\right)$$

Then

$$\mu(x,t) = \sum_{n=1}^{\infty} \left(1 - \cos\frac{n\pi}{2}\right) \sin(n\pi x) \exp\left[-\frac{(2\pi n)^2}{\nu^2} t\right]$$
(c) Steady state means \( \Delta t = 0 \Rightarrow \Delta x = 0 \)

Using B.C. \( d(0) = 0 = 0 \)
\( d(1) = 100 = a \)
\( \Rightarrow d(x) = 100 \times \)

\( (d) \quad \sigma = u + \Delta \quad \sigma_e = u(t) + \Delta E = \Delta E \Rightarrow \sigma_e = 4 \Delta E \times \)

Also \( \sigma(0, t) = u(0, t) + a(0) = 0 \)
\( \sigma(1, t) = u(1, t) + d(1) = 100 \)

and \( \sigma(x, 0) = u(x, 0) + \Delta x = f(x) + \Delta x \)

Thus \( \sigma(x, t) \) solves (2)
Problem 2 (20 pts) Vibrations of a circular membrane of radius 1 with radially symmetric initial shape \( f(r) \) and zero initial velocity are governed by the 2D wave equation,

\[
\begin{aligned}
    u_{rr} &= \Delta u, \quad 0 < r < 1 \text{ and } t > 0, \\
    u(1, t) &= 0, \quad t > 0, \\
    \hat{u}(r, 0) &= f(r), \quad 0 < r < 1, \\
    u_t(r, 0) &= 0, \quad 0 < r < 1.
\end{aligned}
\]  

(3)

A general solution to (3) has the form

\[ u(r, t) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n t) J_0(\alpha_n r), \]

where \( \alpha_n \) is the \( n \)-th positive zero of \( J_0(r) \).

(a) Use the initial conditions and the orthogonality relations for Bessel functions (see formula sheet), to show that

\[ A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 f(r) J_0(\alpha_n r) r \, dr. \]

(b) Solve (3) with \( f(r) = 1 - r^2 \). \textbf{Hint:} Use formula sheet.

(a) \( u(r, t) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n t) J_0(\alpha_n r) \). \textbf{Using initial conditions:}

\[ u(r, 0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) = \text{Bessel series expansion of } f(r) \]

\[ (f(r), J_0(\alpha_n r)) = \left( \sum_{n=1}^{\infty} A_n J_0(\alpha_n r), J_0(\alpha_n r) \right) \]

\[ = A_n \left( J_0(\alpha_n r), J_0(\alpha_n r) \right) \]

\[ \text{Orthogonality relations.} \]

\[ \text{Also: } (J_0(\alpha_n r), J_0(\alpha_n r)) = \frac{1}{2} J_1^2(\alpha_n) \]

\[ \therefore A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 f(r) J_0(\alpha_n r) r \, dr \]
We need to compute:

\[ A_n = \frac{2}{J_1^2(a_n)} \int_0^1 (1-r^2) J_0(a_m r) \, r \, dr \]

\[ = \frac{2}{J_1^2(a_n)} \frac{2}{a_n^2} J_2(a_n) \]

Thus,

\[ u(x, t) = \sum_{m=1}^{\infty} u \frac{J_2(a_m)}{J_1^2(a_m)} \cos(amt) J_0(a_m r) \]
Problem 3 (20 pts) Consider the 2D Laplace equation below which models the steady state temperature distribution of a square plate where all the sides but the top one are dipped in an ice bath,

\[
\begin{align*}
& u_{xx} + u_{yy} = 0, \quad 0 < x < 1 \text{ and } 0 < y < 1, \\
& u(0, y) = u(1, y) = 0, \quad 0 < y < 1, \\
& u(x, 0) = 0, \quad 0 < x < 1, \\
& u(x, 1) = f(x), \quad 0 < x < 1.
\end{align*}
\]

Separation of variables with \(u(x, y) = X(x)Y(y)\) gives

\[
\begin{align*}
& X'' + kX = 0, \quad X(0) = 0, \quad X(1) = 0 \\
& Y'' - kY = 0, \quad Y(0) = 0.
\end{align*}
\]

(a) Assuming \(k = \mu^2 > 0\), obtain the product solutions to (4):

\[u_n(x, y) = B_n \sin(n\pi x) \sinh(n\pi y)\].

(b) Write down the general form of a solution to (4) and express \(B_n\) in terms of \(f(x)\).

(c) Solve (4) with \(f(x) = 100\).

(a) The problem for \(X\) is:

\[
\begin{align*}
& X'' + \mu^2 X = 0 \\
& X(0) = X(1) = 0
\end{align*}
\]

\[
\Rightarrow \quad X_n(x) = \sin(n\pi x)
\]

(b) The problem for \(Y\) is:

\[
\begin{align*}
& Y'' - (n\pi)^2 Y = 0 \\
& Y_n(0) = Y_n(1) = 0
\end{align*}
\]

\[
\Rightarrow \quad Y_n(y) = \sinh(n\pi y)
\]

Product solutions are then:

\[
\begin{align*}
& u_n(x, y) = X_n(x)Y_n(y) = B_n \sin(n\pi x) \sinh(n\pi y).
\end{align*}
\]
(b) A general solution to (4) is:

\[ u(x, y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi x) \sinh(n\pi y) \]

To find \( B_n \) we use B.C.:

\[ u(x, 1) = f(x) = \sum_{n=1}^{\infty} B_n \sinh(n\pi) \sinh(n\pi x) \]

Thus

\[ B_n \sinh(n\pi) = 2 \int_0^1 f(x) \sinh(n\pi x) \, dx \]

\[ B_n = \frac{2}{\sinh(n\pi)} \int_0^1 f(x) \sinh(n\pi x) \, dx \]

(c) For the particular \( f(x) \) we have:

\[ B_n = \frac{2}{\sinh(n\pi)} \int_0^1 \sinh(n\pi x) \, dx = \frac{2}{\sinh(n\pi)} \left[ \frac{\cosh(n\pi x)}{n\pi} \right]_0^1 \]

\[ = \frac{2(1 - (-1)^n)}{(\sinh(n\pi))(n\pi)} \]
Problem 4 (20 pts) Use the Fourier transform method to find the solution $u(x, t)$ of

$$\begin{cases}
u_{tt} + \nu_{xxxx} = 0, & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = f(x), & x \in \mathbb{R}, \\
u_t(x, 0) = 0, & x \in \mathbb{R},
\end{cases}$$

assuming $f(x)$ has a Fourier transform. Give your answer in the form of an inverse Fourier transform.

\[
\begin{align*}
\hat{u}(\omega, t) &= \hat{u}(\omega, 0) e^{-\omega^2 t} + \frac{\partial}{\partial t} \hat{u}(\omega, 0) t e^{-\omega^2 t} \\
&= \hat{u}(\omega, 0) e^{-\omega^2 t} + \frac{\partial}{\partial t} \hat{u}(\omega, 0) t e^{-\omega^2 t} \\
&= A(\omega) e^{-\omega^2 t} + B(\omega) \omega t e^{-\omega^2 t} \\
&= A(\omega) e^{-\omega^2 t} + B(\omega) \omega t e^{-\omega^2 t}
\end{align*}
\]

Thus

\[
\hat{u}(\omega, t) = \hat{f}(\omega) e^{-\omega^2 t}
\]

and

\[
\begin{align*}
u(x, t) &= \frac{1}{2\pi} \int d\omega \ e^{i\omega x} \hat{f}(\omega) e^{-\omega^2 t}
\end{align*}
\]
Problem 5 (15 pts)
(a) Let $a \in \mathbb{R}$. Show that $\mathcal{F}(f(x - a))(\omega) = \exp[-i\omega] \hat{f}(\omega)$.
(b) Show that

$$\frac{1}{2} \mathcal{F}(f(x-a) + f(x+a))(\omega) = \cos(a\omega) \hat{f}(\omega).$$

(a)

$$\mathcal{F}(f(x-a))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ f(x-a) \ e^{-i\omega x}$$

C.o.v.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ f(z) \ e^{-i\omega(z+a)}$$

$$= e^{-i\omega a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ f(z) \ e^{-i\omega z}$$

$$= e^{-i\omega a} \hat{f}(\omega)$$

(b)

$$\frac{1}{2} \left( \mathcal{F}(f(x-a))(\omega) + \mathcal{F}(f(x+a))(\omega) \right)$$

$$= \frac{e^{-i\omega} + e^{i\omega}}{2} \hat{f}(\omega)$$

$$= \cos(a\omega) \hat{f}(\omega)$$
Problem 6 (20 pts)
(a) Let \( c > 0 \). Use the Fourier transform method to show that the solution to the wave equation on an infinite string

\[
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad x \in \mathbb{R} \text{ and } t > 0, \\
u(x, 0) &= f(x), \quad x \in \mathbb{R}, \\
u_t(x, 0) &= 0, \quad x \in \mathbb{R}.
\end{aligned}
\]

is

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \cos(c \omega t) e^{i \omega x} d\omega.
\]

(b) Use (a) and the identity of Problem 5 (b) to show D'Alembert's form of the solution to the wave equation, that is:

\[
u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)].
\]

(c) Explain the physical interpretation of D'Alembert's solution.

(a) Taking \( \mathcal{F} \) of PDE we get:

\[
\frac{d^2 \hat{u}(\omega, t)}{dt^2} = -c^2 \omega^2 \hat{u}(\omega, t)
\]

\[
\Rightarrow \quad \hat{u}(\omega, t) = A(\omega) \cos ct + B(\omega) \sin ct
\]

\[
\hat{u}(\omega, 0) = \hat{f}(\omega) = A(\omega)
\]

\[
\hat{u}_t(\omega, 0) = -A(\omega)(\omega) \sin ct + B(\omega)(\omega) \cos ct
\]

\[
\hat{u}_t(\omega, 0) = B(\omega)(\omega) = 0 \quad \Rightarrow \quad B(\omega) = 0
\]

Thus \( \hat{u}(\omega, t) = \hat{f}(\omega) \cos ct \)

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dw \ e^{i \omega x} \hat{f}(\omega) \cos(c \omega t)
\]

(b) \( \hat{u}(\omega, t) = \cos(c \omega t) \hat{f}(\omega) = \frac{1}{2} \mathcal{F} (f(x - ct) + f(x + ct)) (\omega)
\]

\[
\Rightarrow \quad \nu(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)]
\]
The physical setup predicts that the initial shape of the string creates two waves:

- \( \frac{1}{2} f(x + ct) \) in left going wave
- \( \frac{1}{2} f(x - ct) \) in right going wave

where waves move at a constant speed \( c \).
Problem 7 (10 pts) Let $a > 0$. The function $f(x) = e^{-ax}$ solves the differential equation $f'' - a^2 f + 2a \delta_0(x) = 0$ in the sense of distributions. Deduce that

$$
\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}.
$$

$$
\mathcal{F} \left( f'' - a^2 f + 2a \delta_0 \right)(\omega) = 0
$$

$$
-\omega^2 \hat{f}(\omega) - a^2 \hat{f}(\omega) + \frac{2a}{\sqrt{2\pi}} = 0
$$

$$
\hat{f}(\omega) = \frac{2\omega}{\sqrt{2\pi}} \frac{1}{\omega^2 + a^2} = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2 + a^2}
$$
Problem 8 (20 pts)

(a) Let $a, b > 0$. Show that

$$\mathcal{F}(\exp[-ax^2] * \exp[-bx^2]) (\omega) = \frac{1}{2\sqrt{ab}} \exp \left[-\frac{\omega^2}{4ab} (a + b)\right].$$

(b) Use (a) to show that the convolution of two Gaussian functions is:

$$\exp[-ax^2] * \exp[-bx^2] = \frac{1}{\sqrt{2(a+b)}} \exp \left[-\frac{ab}{a+b} x^2\right].$$

(c) Recall that the solution to heat equation $u_t = u_{xx}$ on the real line with initial temperature $u(x, 0) = f(x)$ is $u(x, t) = (g_t * f)(x)$, where $g_t(x)$ is the heat or Gauss kernel,

$$g_t(x) = \frac{1}{\sqrt{2t}} \exp \left[-\frac{x^2}{4t}\right],$$

which is a Gaussian function of $x$. Show that when the initial temperature is the Gaussian function $f(x) = \exp[-kx^2]$, then $u(x, t)$ is also a Gaussian function of $x$,

$$u(x, t) = \frac{1}{\sqrt{4kt + 1}} \exp \left[-\frac{kx^2}{4kt + 1}\right].$$

\[\text{(a)} \quad \mathcal{F} (e^{-ax^2} * e^{-bx^2})(\omega) = \mathcal{F}(e^{-ax^2})(\omega) \cdot \mathcal{F}(e^{-bx^2})(\omega) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a} \cdot \frac{1}{\sqrt{2b}} e^{-\omega^2/4b} = \frac{1}{2\sqrt{ab}} e^{-\omega^2 \frac{a+b}{ab}} \quad (\ast)\]

\[\text{(b)} \quad (e^{-ax^2} * e^{-bx^2})(x) = \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\omega}} e^{-\omega \frac{a+b}{2ab}} \right) (x) \]

The coefficient multiplying exponent in (\ast) is $\frac{a+b}{ab} = \frac{1}{4a}$

$$\Rightarrow a' = \frac{ab}{a+b}$$

$$\Rightarrow (e^{-ax^2} * e^{-bx^2})(x) = \frac{1}{2 \sqrt{ab}} \sqrt{\frac{2ab}{a+b}} e^{-\frac{ab}{2a} x^2}$$
Simplifying we get result:

\[(e^{-ax^2} * e^{-bx^2})(x) = \frac{1}{\sqrt{2}(a+b)}e^{-\frac{ab}{a+b} x^2}\]

\[(c) \ u(x,t) = (g_t * e^{-kx^2})(x)\]

We need to apply formula from (6) with \(a = \frac{t}{4t}\) and \(b = \frac{t}{4t}\)

\[
\Rightarrow u(x,t) = \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{2(\frac{t}{4t} + k)}} \exp\left[-\frac{k}{4tk+1} x^2\right]
\]

\[
= \frac{1}{\sqrt{4(\frac{1}{4} + k)}} \exp\left[-\frac{k}{4tk+1} x^2\right]
\]

\[
= \frac{1}{\sqrt{1+4kt}} \exp\left[-\frac{k}{4tk+1} x^2\right]
\]