Math 3150-1 PDEs for Engineers – Final Exam
December 15 2008

Name: ______________________ **uNID:** __________

All exercises are independent

Total: 150/150 points

Problem 1 (25 pts) Consider the 1D heat equation with homogeneous Dirichlet boundary conditions,

$$\begin{cases} u_t = 4u_{xx},\ 0 < x < 1 \text{ and } t > 0, \\ u(0,t) = u(1,t) = 0,\ t > 0, \\ u(x,0) = f(x),\ 0 < x < 1. \end{cases} \tag{1}$$

(a) Use separation of variables with $u(x,t) = X(x)T(t)$ to show that a general solution to (1) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\exp[-(2n\pi)^2 t].$$

Specify what are the coefficients b_n in terms of $f(x)$.

(b) Solve (1) with

$$f(x) = \begin{cases} 100 & \text{if } 0 < x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

(c) Find the steady state solution $s(x)$ to

$$\begin{cases} v_t = 4v_{xx},\ 0 < x < 1 \text{ and } t > 0, \\ v(0,t) = 0,\ v(1,t) = 100,\ t > 0, \\ v(x,0) = g(x),\ 0 < x < 1. \end{cases} \tag{2}$$

(d) Solve (2) with $g(x) = f(x) + s(x)$ by using your answer to (b).

(a) Plugging $u(x,t) = X(x)\,T(t)$ into PDE:

$XT' = 4X''T \iff \dfrac{X''}{X} = \dfrac{T'}{4T}$ = constant < 0 to get decay in temp $= -\lambda^2$

$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = X(1) = 0 \end{cases}$ Solving for X we get:

$X(x) = a\cos\lambda x + b\sin\lambda x$

$X(0) = a = 0$

$X(1) = b\sin\lambda = 0 \Rightarrow b = 0$ (trivial sol)

or $\lambda = \lambda_n = n\pi$

$\Rightarrow \boxed{X_n(x) = \sin n\pi x}$

The problem for T is:

$$T_n' = -4\underbrace{(n\pi)^2}_{\lambda^2} T_n \Rightarrow \boxed{T_n(t) = b_n \exp\left[-(2n\pi)^2 t\right]}$$

$\boxed{u_n(x,t) = X_n(x)\,T_n(t)}$ satisfies (1) by construction and so does its sum by the superposition principle:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \exp\left[-(2n\pi)^2 t\right]}$$

Using Initial Conditions:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

b_n = sine series coeff of $f(x)$

$$(f, \sin n\pi x) = b_n (\sin n\pi x, \sin n\pi x); \text{ with } (u,v) = \int_0^1 u(x)\,v(x)\,dx.$$

Besides $(\sin n\pi x, \sin n\pi x) = \int_0^1 \underbrace{\sin^2 n\pi x}_{\text{even}}\,dx = \frac{1}{2}\int_{-1}^{1} \sin^2 n\pi x\,dx = \frac{1}{2}$ ← formula sheet.

$$\Rightarrow \boxed{b_n = 2\int_0^1 f(x)\sin n\pi x\,dx}$$

(b) With the given $f(x)$:

$$b_n = 200\int_0^{1/2} \sin n\pi x\,dx = 200\left.\frac{-\cos n\pi x}{n\pi}\right|_0^{\frac{1}{2}}$$

$$\boxed{= \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \left(1 - \cos\frac{n\pi}{2}\right) \sin[n\pi x] \exp\left[-(2n\pi)^2 t\right]}$$

(c) Steady state means $s_t = 0 \Rightarrow s_{xx} = 0$

$\Rightarrow s(x) = ax + b$

Using B.C. $s(0) = b = 0$, $s(1) = 100 = a$ $\Rightarrow \boxed{s(x) = 100x}$

(d) $v = u + s$ $\quad v_t = u_t + s_t = u_t$, $v_{xx} = u_{xx} + s_{xx} = u_{xx}$ $\Rightarrow \boxed{v_t = 4 v_{xx}}$

Also $v(0,t) = u(0,t) + s(0) = 0$

$v(1,t) = u(1,t) + s(1) = 100$

and $v(x,0) = u(x,0) + s(x) = f(x) + s(x)$

Thus $v(x,t)$ solves (2).

Problem 2 (20 pts) Vibrations of a circular membrane of radius 1 with radially symmetric initial shape $f(r)$ and zero initial velocity are governed by the 2D wave equation,

$$\begin{cases} u_{tt} = \Delta u, \; 0 < r < 1 \text{ and } t > 0, \\ u(1,t) = 0, \; t > 0, \\ u(r,0) = f(r), \; 0 < r < 1, \\ u_t(r,0) = 0, \; 0 < r < 1. \end{cases} \tag{3}$$

A general solution to (3) has the form

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n t) J_0(\alpha_n r),$$

where α_n is the n-th positive zero of $J_0(r)$.

(a) Use the initial conditions and the orthogonality relations for Bessel functions (see formula sheet), to show that

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 f(r) J_0(\alpha_n r) r \, dr.$$

(b) Solve (3) with $f(r) = 1 - r^2$. **Hint:** Use formula sheet.

(a) $u(r,t) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n t) J_0(\alpha_n r)$. Using initial conditions:

$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)$ = Bessel series expansion of $f(r)$

$$(f(r), J_0(\alpha_m r)) = \left(\sum_{n=1}^{\infty} A_n J_0(\alpha_n r), J_0(\alpha_m r)\right)$$

$$= A_m (J_0(\alpha_m r), J_0(\alpha_m r))$$

↑ orthogonality relations. ↓ Here: $(u,v) = \int_0^1 r\,dr\, u(r) v(r)$.

Also: $(J_0(\alpha_m r), J_0(\alpha_m r)) = \frac{1}{2} J_1^2(\alpha_m)$

$$\Rightarrow \boxed{A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 f(r) J_0(\alpha_m r) \, r \, dr}$$

(b) We need to compute:

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1-r^2)\, J_0(\alpha_m r)\, r\, dr$$

$$= \frac{2}{J_1^2(\alpha_n)} \frac{2}{\alpha_n^2} J_2(\alpha_n)$$

$$\Rightarrow \boxed{u(x,t) = \sum_{m=1}^{\infty} \frac{4}{\alpha_m^2} \frac{J_2(\alpha_m)}{J_1^2(\alpha_m)} \cos(\alpha_m t)\, J_0(\alpha_m r)}$$

Problem 3 (20 pts) Consider the 2D Laplace equation below which models the steady state temperature distribution of a square plate where all the sides but the top one are dipped in an ice bath,

$$\begin{cases} u_{xx} + u_{yy} = 0, \ 0 < x < 1 \text{ and } 0 < y < 1, \\ u(0,y) = u(1,y) = 0, \ 0 < y < 1, \\ u(x,0) = 0, \ 0 < x < 1, \\ u(x,1) = f(x), \ 0 < x < 1. \end{cases} \tag{4}$$

Separation of variables with $u(x,y) = X(x)Y(y)$ gives

$$X'' + kX = 0, \ X(0) = 0, \ X(1) = 0$$
$$Y'' - kY = 0, \ Y(0) = 0.$$

(a) Assuming $k = \mu^2 > 0$, obtain the product solutions to (4):
$$u_n(x,y) = B_n \sin(n\pi x)\sinh(n\pi y).$$

(b) Write down the general form of a solution to (4) and express B_n in terms of $f(x)$.

(c) Solve (4) with $f(x) = 100$.

(a) The problem for X is:

$$\begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(1) = 0 \end{cases} \Rightarrow X(x) = a\cos\mu x + b\sin\mu x$$

$X(0) = a = 0$

$X(1) = b\sin\mu = 0 \Rightarrow$ $b = 0$ (trivial sol) or $\mu = \mu_n = n\pi$

$\Rightarrow \boxed{X_n(x) = \sin n\pi x}$

Problem for Y is:

$$\begin{cases} Y_n'' - (n\pi)^2 Y_n = 0 \\ Y_n(0) = 0 \end{cases} \Rightarrow Y_n(y) = a\cosh n\pi y + b\sinh n\pi y$$

$Y_n(0) = a = 0$

$\Rightarrow \boxed{Y_n(y) = b_n \sinh n\pi y}$

Product solutions are then:

$$\boxed{u_n(x,y) = X_n(x)\,Y_n(y) = b_n \sin(n\pi x)\sinh(n\pi y)}$$

(b) A general solution to (4) is:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \sinh(n\pi y)$$

To find B_n we use B.C.

$$u(x,1) = f(x) = \sum_{n=1}^{\infty} \underbrace{B_n \sinh n\pi}_{\text{Fourier sine series coeff of } f(x)} \sin n\pi x$$

$$\Rightarrow B_n \sinh n\pi = 2\int_0^1 f(x) \sin n\pi x \, dx$$

$$\Rightarrow \boxed{B_n = \frac{2}{\sinh n\pi} \int_0^1 f(x) \sin n\pi x \, dx}$$

(c) For the particular $f(x)$ we have:

$$B_n = \frac{200}{\sinh n\pi} \int_0^1 \sin n\pi x \, dx = \frac{200}{\sinh n\pi} \left. \frac{-\cos n\pi x}{n\pi} \right|_0^1$$

$$\boxed{= \frac{200\left(1-(-1)^n\right)}{(\sinh n\pi)(n\pi)}}$$

Problem 4 (20 pts) Use the Fourier transform method to find the solution $u(x,t)$ of

$$\begin{cases} u_{tt} + u_{xxxx} = 0, \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) = f(x), \ x \in \mathbb{R}, \\ u_t(x,0) = 0, \ x \in \mathbb{R}, \end{cases}$$

assuming $f(x)$ has a Fourier transform. Give your answer in the form of an inverse Fourier transform.

$\downarrow \mathcal{F}$

$$\frac{d^2}{dt^2} \hat{u}(\omega,t) + \omega^4 \hat{u}(\omega,t) = 0$$

$$\hat{u}(\omega,t) = A(\omega)\cos(\omega^2 t) + B(\omega)\sin(\omega^2 t)$$

$$\Rightarrow \hat{u}_t(\omega,t) = -A(\omega)\omega^2 \sin(\omega^2 t) + B(\omega)\omega^2 \cos(\omega^2 t)$$

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

$$\hat{u}_t(\omega,0) = B(\omega)\omega^2 = 0 \Rightarrow B(\omega) = 0$$

Thus $\hat{u}(\omega,t) = \hat{f}(\omega)\cos(\omega^2 t)$

and $\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \int d\omega \, e^{i\omega x} \hat{f}(\omega) \cos(\omega^2 t)}$

Problem 5 (15 pts)

(a) Let $a \in \mathbb{R}$. Show that $\mathscr{F}(f(x-a))(\omega) = \exp[-ia\omega]\hat{f}(\omega)$.

(b) Show that

$$\frac{1}{2}\mathscr{F}(f(x-a)+f(x+a))(\omega) = \cos(a\omega)\hat{f}(\omega).$$

(a) $$\mathcal{F}(f(x-a))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x-a)\, e^{-ix\omega}$$

C.o.v

$z = x-a$

$dz = dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, f(z)\, e^{-i(z+a)\omega}$$

$$= e^{-ia\omega}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, f(z)\, e^{-iz\omega}}_{\hat{f}(\omega)}.$$

$$\boxed{= e^{-ia\omega}\hat{f}(\omega)}$$

(b)

$$\frac{1}{2}\left(\mathcal{F}(f(x-a))(\omega) + \mathcal{F}(f(x+a))(\omega)\right)$$

$$= \frac{e^{-ia\omega}+e^{ia\omega}}{2}\hat{f}(\omega)$$

$$\boxed{= \cos(a\omega)\hat{f}(\omega)}$$

Problem 6 (20 pts)

(a) Let $c > 0$. Use the Fourier transform method to show that the solution to the wave equation on an infinite string

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x), \ x \in \mathbb{R}, \\ u_t(x,0) = 0, \ x \in \mathbb{R}. \end{cases}$$

is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\cos(c\omega t)e^{i\omega x}\,d\omega.$$

(b) Use (a) and the identity of Problem 5 (b) to show D'Alembert's form of the solution to the wave equation, that is:

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)].$$

(c) Explain the physical interpretation of D'Alembert's solution.

(a) Taking $\mathcal{F}$ of PDE we get:

$$\frac{d^2}{dt^2}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$$

$$\Rightarrow \quad \hat{u}(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

$$\hat{u}(\omega,0) = \hat{f}(\omega) = A(\omega)$$

$$\hat{u}_t(\omega,t) = -A(\omega)(c\omega)\sin c\omega t + B(\omega)(c\omega)\cos c\omega t$$

$$\hat{u}_t(\omega,0) = B(\omega)(c\omega) = 0 \quad \Rightarrow B(\omega) = 0$$

Thus $\hat{u}(\omega,t) = \hat{f}(\omega)\cos(c\omega t)$

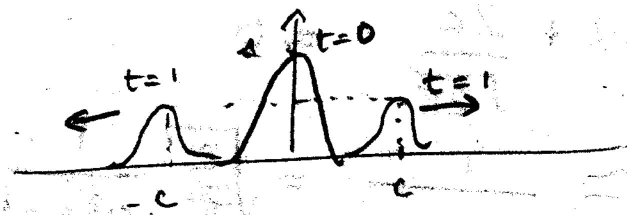
$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{i\omega x}\hat{f}(\omega)\cos(c\omega t)}$$

(b) $\hat{u}(\omega,t) = \cos(c\omega t)\hat{f}(\omega) = \frac{1}{2}\mathcal{F}(f(x-ct) + f(x+ct))(\omega)$

$$\Rightarrow u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$$

(c) The physical interpretation is that the initial shape of the string creates two waves:

- $\frac{1}{2} f(x+ct)$ is left going wave
- $\frac{1}{2} f(x-ct)$ " right " "



where waves move at a constant speed c.

Problem 7 (10 pts) Let $a > 0$. The function $f(x) = e^{-a|x|}$ solves the differential equation $f'' - a^2 f + 2a\delta_0(x) = 0$ in the sense of distributions. Deduce that

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}.$$

$$\mathcal{F}(f'' - a^2 f + 2a\delta_0)(\omega) = 0$$

$$-\omega^2 \widehat{f}(\omega) - a^2 \widehat{f}(\omega) + \frac{2a}{\sqrt{2\pi}} = 0$$

$$\boxed{\widehat{f}(\omega) = \frac{2a}{\sqrt{2\pi}} \frac{1}{\omega^2 + a^2} = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2 + a^2}}$$

Problem 8 (20 pts)

(a) Let $a, b > 0$. Show that

$$\mathcal{F}(\exp[-ax^2] * \exp[-bx^2])(\omega) = \frac{1}{2\sqrt{ab}} \exp\left[-\frac{\omega^2}{4ab}(a+b)\right].$$

(b) Use (a) to show that the convolution of two Gaussian functions is:

$$\exp[-ax^2] * \exp[-bx^2] = \frac{1}{\sqrt{2(a+b)}} \exp\left[-\frac{ab}{a+b}x^2\right].$$

(c) Recall that the solution to heat equation $u_t = u_{xx}$ on the real line with initial temperature $u(x,0) = f(x)$ is $u(x,t) = (g_t * f)(x)$, where $g_t(x)$ is the heat or Gauss kernel,

$$g_t(x) = \frac{1}{\sqrt{2t}} \exp\left[-\frac{x^2}{4t}\right],$$

which is a Gaussian function of x. Show that when the initial temperature is the Gaussian function $f(x) = \exp[-kx^2]$, then $u(x,t)$ is also a Gaussian function of x,

$$u(x,t) = \frac{1}{\sqrt{4kt+1}} \exp\left[-\frac{kx^2}{4kt+1}\right].$$

(a) $\mathcal{F}(e^{-ax^2} * e^{-bx^2})(\omega) = \mathcal{F}(e^{-ax^2})(\omega) \times \mathcal{F}(e^{-bx^2})(\omega)$

$$= \frac{1}{\sqrt{2a}} e^{-\omega^2/4a} \cdot \frac{1}{\sqrt{2b}} e^{-\omega^2/4b}$$

$$= \frac{1}{2\sqrt{ab}} e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$= \frac{1}{2\sqrt{ab}} e^{-\omega^2 \frac{a+b}{ab}} \quad (*)$$

(b) $(e^{-ax^2} * e^{-bx^2})(x) = \mathcal{F}^{-1}(\quad)(x)$

The coefficient multiplying exponent in $(*)$ is $\frac{a+b}{ab} = \frac{1}{4a'}$

$$\Rightarrow \quad a' = \frac{ab}{a+b}$$

$$\Rightarrow (e^{-ax^2} * e^{-bx^2})(x) = \frac{1}{2\sqrt{ab}} \underbrace{\sqrt{2\frac{ab}{a+b}}}_{\sqrt{2a'}} \, e^{-\underbrace{\frac{ab}{a+b}}_{a'} x^2}$$

Simplifying we get result:

$$\left(e^{-ax^2} * e^{-bx^2}\right)(x) = \frac{1}{\sqrt{2(a+b)}} e^{-\frac{ab}{a+b}x^2}$$

(c) $u(x,t) = \left(g_t * e^{-kx^2}\right)(x)$. We need to apply formula from (b) with $a = \frac{1}{4t}$ and $b = k$.

from g_t from $\frac{1}{\sqrt{2(a+b)}}$ $\Rightarrow \frac{ab}{a+b} = \frac{k/4t}{1/4t + k} = \frac{k}{4tk+1}$

$$\Rightarrow \boxed{u(x,t)} = \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{2\left(\frac{1}{4t}+k\right)}} \exp\left[-\frac{k}{4tk+1}x^2\right]$$

$$= \underbrace{\frac{1}{\sqrt{4\left(\frac{1}{4}+kt\right)}}}_{= \frac{1}{\sqrt{1+4kt}}} \exp\left[-\frac{k}{4tk+1}x^2\right]$$

$$= \boxed{\frac{1}{\sqrt{1+4kt}} \exp\left[-\frac{k}{4tk+1}x^2\right]}$$