Math 3150-1, Practice Final
December 10 2008

Problem 1 (25 pts) Consider the 1D heat equation with homogeneous Neumann boundary conditions modeling a bar with insulated ends:

$$\begin{cases} u_t = u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u_x(0,t) = u_x(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x), & \text{for } 0 < x < 1. \end{cases} \tag{1}$$

(a) Use separation of variables with $u(x,t) = X(x)T(t)$ to show that a general solution to (1) is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp[-(n\pi)^2 t].$$

Specify what the coefficients a_n, $n = 0, 1, 2, \ldots$ are in terms of $f(x)$.

(b) Solve (1) with $f(x) = 100x$.

(c) Now consider the following 1D heat equation with *inhomogeneous* Neumann boundary conditions:

$$\begin{cases} v_t = v_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ v_x(0,t) = v_x(1,t) = 1 & \text{for } t > 0, \\ v(x,0) = g(x), & \text{for } 0 < x < 1 \end{cases} \tag{2}$$

Show that $v(x,t) = u(x,t) + x$ solves (2) with $g(x) = f(x) + x$ and $u(x,t)$ as in (b).

(a) Plugging ansatz into (1) we get

$$XT' = X''T \Rightarrow \underbrace{\frac{X''}{X}}_{\text{dep on } x} = \underbrace{\frac{T'}{T}}_{\text{dep on } t} = \text{constant} \Rightarrow \begin{cases} X'' = -\lambda^2 X, & X'(0) = X'(1) = 0 \\ T' = -\lambda^2 T \end{cases}$$

negative to reproduce physical decay of temp.

$T(t) = a \exp[-\lambda^2 t]$

$X(x) = a\cos(\lambda x) + b\sin(\lambda x)$

$X'(x) = -a\lambda \sin\lambda x + b\lambda\cos(\lambda x)$

$X'(0) = b\lambda = 0 \Rightarrow \begin{cases} b = 0 \\ \text{or} \\ \lambda = 0 \end{cases}$

<u>when $b=0$</u>: $X'(1) = -a\lambda\sin\lambda = 0$

$\Rightarrow \lambda = \lambda_n = n\pi,\ n = 1, 2, \ldots$

Thus:

$$\begin{cases} X_n(x) = a_n \cos(n\pi x), & n \geq 1 \\ T_n(x) = \exp[-(n\pi)^2 t] \end{cases} \qquad \begin{cases} X_0(x) = a_0 \\ T_0(x) = 1 \end{cases}$$

The product solutions $u_n(x,t) = X_n(x)T_n(t)$ solve (1) and so does their sum

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp(-(n\pi)^2 t)}$$

Since $u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = f(x)$,

$\Rightarrow a_n =$ cosine series coeff of $f(x)$: $\boxed{a_0 = \int_0^1 f(x)\,dx;\ a_n = 2\int_0^1 f(x)\cos n\pi x\,dx}$

(b) $\boxed{a_0 = \int_0^1 100x\,dx = 100 \frac{x^2}{2}\Big|_0^1 = 50}$

$\boxed{a_n} = 2\int_0^1 100x \cos(n\pi x)\,dx = 200x \frac{\sin(n\pi x)}{n\pi}\Big|_0^1 - 200\int_0^1 \frac{\sin n\pi x}{n\pi}\,dx$

$= 200 \frac{\cos n\pi x}{(n\pi)^2}\Big|_0^1 = \boxed{\frac{200}{(n\pi)^2}((-1)^n - 1)}$

(c) $v_t = u_t + \cancel{\frac{\partial x}{\partial t}}^{0} = u_t$; $v_{xx} = u_{xx} + \cancel{\frac{\partial^2 x}{\partial x^2}}^{0} = u_{xx} \Rightarrow \boxed{v_t = v_{xx}}$

$v_x(x,t) = u_x(x,t) + 1 \Rightarrow \boxed{\begin{aligned} v_x(0,t) &= \cancel{u_x(0,t)} + 1 = 1 \\ v_x(1,t) &= \cancel{u_x(1,t)} + 1 = 1 \end{aligned}}$

$\boxed{v(x,0)} = u(x,0) + x = \boxed{f(x) + x}$ (Thus $v(x,t)$ solves (2))

Problem 2 (25 pts) Consider a circular plate of radius 1 with radially symmetric initial temperature distribution $f(r)$, and where the outer rim is kept in an ice bath. The temperature distribution $u(r,t)$ is also radially symmetric and satisfies the 2D heat equation,

$$\begin{cases} u_t = \Delta u \ \text{ for } 0 < r < 1 \text{ and } t > 0, \\ u(r,0) = f(r) \ \text{ for } 0 < r < 1, \\ u(1,t) = 0 \ \text{ for } t > 0. \end{cases} \tag{3}$$

A general solution to (3) has the form,

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \exp[-\alpha_n^2 t],$$

where α_n is the n-th positive zero of $J_0(r)$.

(a) Use the initial conditions and the orthogonality conditions for Bessel functions (see end of exam), to show that

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 f(r) J_0(\alpha_n r) r\,dr.$$

(b) Solve (3) with initial temperature $f(r) = J_0(\alpha_2 r)$.

(c) Show that $\Delta(r\cos\theta) = 0$. (Easier in Cartesian coordinates).

(d) Show that $v(r,\theta,t) = u(r,t) + r\cos\theta$ solves the following 2D heat equation with *inhomogeneous* Dirichlet boundary conditions,

$$\begin{cases} v_t = \Delta v \ \text{ for } 0 < r < 1 \text{ and } t > 0, \\ v(r,0) = f(r) + r\cos\theta \ \text{ for } 0 < r < 1, \\ v(1,t) = \cos\theta \ \text{ for } t > 0. \end{cases}$$

(a) $u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) = f(r)$

$\hookrightarrow$ Bessel series coeff of $f(r)$.

$\Rightarrow \left(f(r), J_0(\alpha_m r)\right) = \left(\sum_{n=1}^{\infty} A_n J_0(\alpha_n r), J_0(\alpha_m r)\right)$

$\Rightarrow \boxed{A_m = \frac{(J_0(\alpha_m r), f(r))}{(J_0(\alpha_m r), J_0(\alpha_m r))} = \frac{2}{J_1^2(\alpha_m)} \int_0^1 f(r) J_0(\alpha_m r)\, r\, dr}$

(b) $f(r)$ is already given in Bessel series form $\therefore A_2 = 1$, $A_n = 0$ for $n \neq 2$.

$\Rightarrow \boxed{u(r,t) = J_0(\alpha_2 r) \exp\left[-\alpha_2^2 t\right]}$

(c) $r\cos\theta = x$ and $\frac{\partial^2 x}{\partial x^2} = 0 \Rightarrow \Delta(x) = \Delta(r\cos\theta) = 0$.

(d)
$$\left.\begin{aligned} v_t &= u_t + \underbrace{\frac{\partial}{\partial r}(r\cos\theta)}_{=0} = u_t \\ \Delta v &= \Delta u + \underbrace{\Delta(r\cos\theta)}_{=0} = \Delta u \end{aligned}\right\} \Rightarrow \boxed{\Delta v = v_t}.$$

$\boxed{v(r,0) = u(r,0) + r\cos\theta = f(r) + r\cos\theta}$

$\boxed{v(1,t) = u(1,t) + 1\cdot\cos\theta = \cos\theta}$

$\Rightarrow$ v solves desired equation.

Problem 3 (25 pts) Consider the 2D wave equation below which models the vibrations of square membrane with fixed edges, initial position $f(x,y)$ and zero initial velocity.

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } t > 0, \\ u(0,y,t) = u(1,y,t) = 0, & \text{for } 0 < y < 1 \text{ and } t > 0 \\ u(x,0,t) = u(x,1,t) = 0, & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(x,y,0) = f(x,y), & \text{for } 0 < x < 1,\ 0 < y < 1 \\ u_t(x,y,0) = 0, & \text{for } 0 < x < 1,\ 0 < y < 1. \end{cases} \tag{4}$$

Separation of variables with $u(x,y,t) = X(x)Y(y)T(t)$ gives the ODEs:

$$X'' + \mu^2 X = 0,\ X(0) = 0,\ X(1) = 0$$
$$Y'' + \nu^2 Y = 0,\ Y(0) = 0,\ Y(1) = 0$$
$$T'' + (\mu^2 + \nu^2)T = 0,\ T'(0) = 0.$$

(a) Obtain the product solutions

$$u_{m,n}(x,y,t) = B_{m,n}\cos(\lambda_{m,n}t)\sin(m\pi x)\sin(n\pi y).$$

where $\lambda_{m,n} = \sqrt{(m\pi)^2 + (n\pi)^2}$. **Note:** The ODE's for X and Y are very similar. Solving one of them in detail and stating the result for the other one should be enough.

(b) Write down the general form of a solution $u(x,y,t)$ to (4). Use initial conditions and orthogonality of double sine series to express $B_{m,n}$ in terms of $f(x,y)$.

(c) Using that

$$\int_0^1 x(1-x)\sin(m\pi x)dx = \frac{2((-1)^m - 1)}{\pi^3 m^3},$$

find the coefficients $B_{m,n}$ in the double sine series,

$$x(1-x)y(1-y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{m,n}\sin(m\pi x)\sin(n\pi y).$$

(d) Solve 2D wave equation (4) with $f(x,y) = x(1-x)y(1-y)$.

(a) $\begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(1) = 0 \end{cases} \Rightarrow X(x) = a\cos\mu x + b\sin\mu x$

$X(0) = a = 0$

$X(1) = b\sin\mu = 0 \Rightarrow \mu \to \mu_n = n\pi$

$n = 1, 2, \dots$

$\Rightarrow X_n(x) = b_n \sin n\pi x$

$Y_m(y) = b_m \sin m\pi y$ (similarly)

$\Rightarrow T''_{mn} = -\lambda_{mn}^2 T_{m,n} \Rightarrow T_{m,n}(t) = a_{mn}\cos\lambda_{mn}t + b_{mn}\sin\lambda_{mn}t$

$T'_{m,n}(t) = -\lambda_{mn}a_n\sin\lambda_{mn}t + b_{mn}\lambda_{mn}\cos(\lambda_{mn}t)$

$T'_{mn}(0) = b_{mn}\lambda_{mn} = 0 \Rightarrow b_{mn} = 0$

$\Rightarrow$ product solutions

$$\boxed{u_{mn}(x,y,t) = B_{m,n}\cos(\lambda_{m,n}t)\sin(m\pi x)\sin(n\pi y)}$$

(b) $u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \cos(\lambda_{mn} t) \sin(m\pi x) \sin(n\pi y)$

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin(m\pi x) \sin(n\pi y)$$

B_{mn} = coeff. in double sine series of $f(x,y)$

$$\Rightarrow \boxed{B_{mn}} = \frac{(f(x,y), \sin(m\pi x)\sin(n\pi y))}{(\sin(m\pi x)\sin(n\pi y), \sin(m\pi x)\sin(n\pi y))}$$

$$= \boxed{4 \int_0^1 \int_0^1 f(x,y) \sin(m\pi x) \sin(n\pi y)\, dx\, dy}$$

(c) $\boxed{B_{mn}} = \int_0^1 dx \int_0^1 dy \; x(1-x) y(1-y) \sin(m\pi x) \sin(n\pi y)$

$$= \left(\int_0^1 dx\, x(1-x) \sin(m\pi x)\right)\left(\int_0^1 dy\, y(1-y) \sin(n\pi y)\right)$$

$$= \frac{2((-1)^m - 1)}{\pi^3 m^3} \times \frac{2((-1)^n - 1)}{\pi^3 n^3}$$

$$= \boxed{\frac{4}{\pi^6 n^3 m^3}((-1)^m - 1)((-1)^n - 1)}$$

(d) Solution is given by

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \cos(\lambda_{mn} t) \sin(m\pi x) \sin(n\pi y)$$

with B_{mn} as in (c).

Problem 4 (25 pts) Use the Fourier transform method to solve

$$\begin{cases} u_{tt} = u_{txx}, & \text{for } x \in \mathbb{R} \text{ and } t > 0. \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R} \\ u_t(x,0) = g(x), & \text{for } x \in \mathbb{R} \end{cases}$$

where $f(x)$ and $g(x)$ have Fourier transforms. Give your answer in the form of an inverse Fourier transform.

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$$\begin{cases} \frac{d^2}{dt^2}\hat{u}(\omega,t) = \frac{d}{dt}\left(-\omega^2 \hat{u}(\omega,t)\right) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \\ \frac{d\hat{u}}{dt}(\omega,0) = \hat{g}(\omega) \end{cases}$$

Solving first eq gives:

$$\frac{d\hat{u}(\omega,t)}{dt} = C(\omega)e^{-\omega^2 t}$$

$$\frac{d\hat{u}(\omega,0)}{dt} = C(\omega) = \hat{g}(\omega)$$

$$\Rightarrow \boxed{\frac{d\hat{u}(\omega,t)}{dt} = \hat{g}(\omega)\, e^{-\omega^2 t}}$$

integration in t

$$\Rightarrow \hat{u}(\omega,t) = \hat{g}(\omega)\frac{e^{-\omega^2 t}}{-\omega^2} + D(\omega)$$

$$\hat{u}(\omega,0) = \frac{\hat{g}(\omega)}{-\omega^2} + D(\omega) = \hat{f}(\omega)$$

$$\Rightarrow \boxed{\hat{u}(\omega,t) = \hat{f}(\omega) - \frac{\hat{g}(\omega)}{\omega^2}\left[e^{-\omega^2 t} + 1\right]}$$

Problem 5 (25 pts) Let $f(x) = x\exp[-x^2/2]$ and $g(x) = \exp[-x^2]$.

(a) What are the Fourier transforms of f and g?

(b) What is the Fourier transform of $f * g$?

(c) Use (b) and operational properties of the Fourier Transform to show that

$$(f * g)(x) = \frac{2}{3\sqrt{3}} x \exp[-x^2/3].$$

(a) Using table $\boxed{\hat{f}(\omega) = \mathcal{F}(x e^{-x^2/2})(\omega) = i\frac{d}{d\omega}\mathcal{F}(e^{-x^2/2})(\omega) = -i\omega e^{-\omega^2/2}}$

$$\boxed{\hat{g}(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}}$$

(b) $\boxed{\mathcal{F}(f*g)(\omega) = \hat{f}(\omega)\hat{g}(\omega) = -\frac{i\omega}{\sqrt{2}} e^{-3\omega^2/4}}$

(c)

$$\mathcal{F}(e^{-x^2/3})(\omega) = \sqrt{\tfrac{3}{2}}\, e^{-3\omega^2/4}$$

$$\mathcal{F}(x e^{-x^2/3})(\omega) = -i\left(\tfrac{3}{2}\right)^{\frac{3}{2}} e^{-3\omega^2/4}$$

$$\mathcal{F}\left(\tfrac{2}{3\sqrt{3}} x e^{-x^2/3}\right)(\omega) = -\frac{i\omega}{\sqrt{2}} e^{-3\omega^2/4}$$

$$\Rightarrow \boxed{(f*g)(x) = \mathcal{F}^{-1}\left(-\frac{i\omega}{\sqrt{2}} e^{-3\omega^2/4}\right)(x) = \frac{2}{3\sqrt{3}} x e^{-x^2/3}}$$

Problem 6 (25 pts) Consider the heat equation on an infinite rod with non constant coefficients

$$\begin{cases} u_t = t^2 u_{xx}, & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

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(a) Use the Fourier transform method to show that the solution satisfies $\widehat{u}(\omega, t) = \widehat{f}(\omega)\exp[-\omega^2 t^3/3]$.
Hint: Solutions ot the ODE $y' + ax^2 y = 0$ have the form, $y(x) = C\exp[-ax^3/3]$.

(b) Express $u(x,t)$ as a convolution.

(a) $\frac{d}{dt}\hat{u}(\omega,t) = -t^2\omega^2\hat{u}(\omega,t)$

$\Rightarrow \hat{u}(\omega,t) = C(\omega)e^{-\frac{t^3\omega^2}{3}}$ (using Hint)

$\hat{u}(\omega,0) = C(\omega) = \hat{f}(\omega)$

$\Rightarrow \boxed{\hat{u}(\omega,t) = \hat{f}(\omega)\exp[-t^3\omega^2/3]}$

(b) we need to find $g_t(x)$ s.t. $\hat{g}_t(\omega) = \exp[-t^3\omega^2/3]$

then $\hat{u}(\omega,t) = \hat{f}(\omega)\,\hat{g}_t(\omega) \Rightarrow \boxed{u(x,t) = (f * g_t)(x)}$

Using table:

$$\frac{2t^{3/2}}{\sqrt{6}}e^{-(t^3/3)\omega^2} = \mathcal{F}(e^{-\frac{3}{4t^3}x^2})(\omega) \Rightarrow \boxed{g_t(x) = \mathcal{F}^{-1}(e^{-t^3\omega^2/3})(x) = \frac{\sqrt{6}}{2t^{3/2}}\exp[-3x^2/(4t^3)]}$$

Problem 7 (25 pts) Let

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

then f satisfies the differential equation in the sense of generalized functions: $f + f' = \delta_0$. Fourier transform this expression and use operational properties of the Fourier transform to show that

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1-i\omega}{1+\omega^2}.$$

$f + f' = \delta_0$

$\hat{f}(\omega) + i\omega\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} = \hat{\delta_0}$

$\hat{f}(\omega)(1+i\omega) = \frac{1}{\sqrt{2\pi}}$

$$\Rightarrow \boxed{\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\omega} = \frac{1}{\sqrt{2\pi}}\frac{1-i\omega}{(1+i\omega)(1-i\omega)} = \frac{1}{\sqrt{2\pi}}\frac{1-i\omega}{1+\omega^2}}$$