Math 3150-1, Practice Final
December 10 2008

Problem 1 (25 pts) Consider the 1D heat equation with homogeneous Neumann boundary conditions modeling a bar with insulated ends:

$$\begin{cases} u_t = u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u_x(0,t) = u_x(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x), & \text{for } 0 < x < 1. \end{cases} \tag{1}$$

(a) Use separation of variables with $u(x,t) = X(x)T(t)$ to show that a general solution to (1) is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp[-(n\pi)^2 t].$$

Specify what the coefficients a_n, $n = 0, 1, 2, \ldots$ are in terms of $f(x)$.

(b) Solve (1) with $f(x) = 100x$.

(c) Now consider the following 1D heat equation with *inhomogeneous* Neumann boundary conditions:

$$\begin{cases} v_t = v_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ v_x(0,t) = v_x(1,t) = 1 & \text{for } t > 0, \\ v(x,0) = g(x), & \text{for } 0 < x < 1 \end{cases} \tag{2}$$

Show that $v(x,t) = u(x,t) + x$ solves (2) with $g(x) = f(x) + x$ and $u(x,t)$ as in (b).

Problem 2 (25 pts) Consider a circular plate of radius 1 with radially symmetric initial temperature distribution $f(r)$, and where the outer rim is kept in an ice bath. The temperature distribution $u(r,t)$ is also radially symmetric and satisfies the 2D heat equation,

$$\begin{cases} u_t = \Delta u & \text{for } 0 < r < 1 \text{ and } t > 0, \\ u(r,0) = f(r) & \text{for } 0 < r < 1, \\ u(1,t) = 0 & \text{for } t > 0. \end{cases} \tag{3}$$

A general solution to (3) has the form,

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \exp[-\alpha_n^2 t],$$

where α_n is the $n-$th positive zero of $J_0(r)$.

(a) Use the initial conditions and the orthogonality conditions for Bessel functions (see end of exam), to show that

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 f(r) J_0(\alpha_n r) r dr.$$

(b) Solve (3) with initial temperature $f(r) = J_0(\alpha_2 r)$.

(c) Show that $\Delta(r\cos\theta) = 0$. (Easier in Cartesian coordinates).

(d) Show that $v(r,\theta,t) = u(r,t) + r\cos\theta$ solves the following 2D heat equation with *inhomogeneous* Dirichlet boundary conditions,

$$\begin{cases} v_t = \Delta v & \text{for } 0 < r < 1 \text{ and } t > 0, \\ v(r,0) = f(r) + r\cos\theta & \text{for } 0 < r < 1, \\ v(1,t) = \cos\theta & \text{for } t > 0. \end{cases}$$

Problem 3 (25 pts) Consider the 2D wave equation below which models the vibrations of square membrane with fixed edges, initial position $f(x,y)$ and zero initial velocity.

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } t > 0, \\ u(0,y,t) = u(1,y,t) = 0, & \text{for } 0 < y < 1 \text{ and } t > 0 \\ u(x,0,t) = u(x,1,t) = 0, & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(x,y,0) = f(x,y), & \text{for } 0 < x < 1,\ 0 < y < 1 \\ u_t(x,y,0) = 0, & \text{for } 0 < x < 1,\ 0 < y < 1. \end{cases} \tag{4}$$

Separation of variables with $u(x,y,t) = X(x)Y(y)T(t)$ gives the ODEs:

$$X'' + \mu^2 X = 0,\ X(0) = 0,\ X(1) = 0$$

$$Y'' + \nu^2 Y = 0,\ Y(0) = 0,\ Y(1) = 0$$

$$T'' + (\mu^2 + \nu^2)T = 0,\ T'(0) = 0.$$

(a) Obtain the product solutions

$$u_{m,n}(x,y,t) = B_{m,n}\cos(\lambda_{m,n}t)\sin(m\pi x)\sin(n\pi y).$$

where $\lambda_{m,n} = \sqrt{(m\pi)^2 + (n\pi)^2}$. **Note:** The ODE's for X and Y are very similar. Solving one of them in detail and stating the result for the other one should be enough.

(b) Write down the general form of a solution $u(x,y,t)$ to (4). Use initial conditions and orthogonality of double sine series to express $B_{m,n}$ in terms of $f(x,y)$.

(c) Using that

$$\int_0^1 x(1-x)\sin(m\pi x)dx = \frac{2((-1)^m - 1)}{\pi^3 m^3},$$

find the coefficients $B_{m,n}$ in the double sine series,

$$x(1-x)y(1-y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{m,n}\sin(m\pi x)\sin(n\pi y).$$

(d) Solve 2D wave equation (4) with $f(x,y) = x(1-x)y(1-y)$.

Problem 4 (25 pts) Use the Fourier transform method to solve

$$\begin{cases} u_{tt} = u_{txx}, & \text{for } x \in \mathbb{R} \text{ and } t > 0. \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R} \\ u_t(x,0) = g(x), & \text{for } x \in \mathbb{R} \end{cases}$$

where $f(x)$ and $g(x)$ have Fourier transforms. Give your answer in the form of an inverse Fourier transform.

Problem 5 (25 pts) Let $f(x) = x\exp[-x^2/2]$ and $g(x) = \exp[-x^2]$.

(a) What are the Fourier transforms of f and g?

(b) What is the Fourier transform of $f * g$?

(c) Use (b) and operational properties of the Fourier Transform to show that

$$(f * g)(x) = \frac{2}{3\sqrt{3}}x\exp[-x^2/3].$$

Problem 6 (25 pts) Consider the heat equation on an infinite rod with non constant coefficients

$$\begin{cases} u_t = t^2 u_{xx}, & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

(a) Use the Fourier transform method to show that the solution satisfies $\widehat{u}(\omega, t) = \widehat{f}(\omega) \exp[-\omega^2 t^3/3]$.
Hint: Solutions ot the ODE $y' + ax^2 y = 0$ have the form, $y(x) = C \exp[-ax^3/3]$.

(b) Express $u(x,t)$ as a convolution.

Problem 7 (25 pts) Let

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

then f satisfies the differential equation in the sense of generalized functions: $f + f' = \delta_0$. Fourier transform this expression and use operational properties of the Fourier transform to show that

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1 - i\omega}{1 + \omega^2}.$$