MATH 3150-1, PRACTICE MIDTERM EXAM 3
NOVEMBER 26 2008

Total points: 115/115.
Problem 1 (20 pts) Let $f(\theta) = (\pi - \theta)/2$.
(a) Find the Fourier series expansion of $f(\theta)$.
(b) Solve the Dirichlet problem on the unit disk,

$$\begin{cases} \Delta u = 0, \quad 0 < r < 1, \quad 0 < \theta < 2\pi \\ u(1,\theta) = f(\theta), \quad 0 < \theta < 2\pi. \end{cases}$$

Problem 2 (30 pts)
(a) Show that

$$\mathcal{F}(f(ax))(\omega) = \frac{1}{|a|}\mathcal{F}(f)\left(\frac{\omega}{a}\right), \quad a \neq 0.$$

(b) Use (a) to find the Fourier transform of $g(x) = e^{-5|x|}$ from that of $f(x) = e^{-|x|}$.
(c) Use (a) to find the Fourier transform of $g(x) = e^{-ax^2/2}$ from that of $f(x) = e^{-x^2/2}$.
Problem 3 (10 pts) Compute the Fourier transform of

$$f(x) = \begin{cases} x & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4 (20 pts) Use the Fourier transform method to solve

$$\begin{cases} u_{tt}(x,t) = u_{xxxx}(x,t), & x \in \mathbb{R} \text{ and } t > 0 \\ u(x,0) = f(x), \quad x \in \mathbb{R} \\ u_t(x,0) = 0, \quad x \in \mathbb{R}. \end{cases}$$

where

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 5 (20 pts) Use the Fourier transform method to solve

$$\begin{cases} u_t = c^2 u_{xx} + k u_x \\ u(x,0) = f(x) \end{cases}$$

where $k > 0$.
Problem 6 (15 pts) The Poisson kernel is given by

$$P_y(x) = \sqrt{\frac{2}{\pi}}\frac{y}{x^2 + y^2}.$$

(a) Show that $\widehat{P_y}(\omega) = e^{-y|\omega|}$.
(b) Use the Fourier transform to show the semi-group property of the Poisson kernel,

$$(P_y * P_z)(x) = P_{y+z}(x).$$

Useful formulas

0.1. Complex numbers. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. If $z = a + ib$, where $a = \text{Re}\, z$ and $b = \text{Im}\, z$, then $\bar{z} = a - ib$ and $|z|^2 = \bar{z}z = a^2 + b^2$.

0.2. Fourier transforms.

$f(x)$	$\widehat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} dx\, f(x) e^{-i\omega x}$
$\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} d\omega\, \widehat{f}(\omega) e^{i\omega x}$	$\widehat{f}(\omega)$
$\begin{cases} 1 & \text{if } \lvert x\rvert \le 1 \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}} \dfrac{\sin a\omega}{\omega}$
$\dfrac{1}{a^2 + x^2},\ a > 0$	$\sqrt{\dfrac{\pi}{2}} \dfrac{e^{-a\lvert\omega\rvert}}{a}$
$e^{-a\lvert x\rvert},\ a > 0$	$\sqrt{\dfrac{2}{\pi}} \dfrac{a}{a^2 + \omega^2}$
$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f^{(n)}(x)$	$(i\omega)^n \widehat{f}(\omega)$
$x^n f(x)$	$i^n \dfrac{d^n}{d\omega^n} \widehat{f}(\omega)$
$(f * g)(x)$	$\widehat{f}(\omega)\widehat{g}(\omega)$

0.3. Fourier series. For a $2p$ periodic function piecewise smooth function f,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n x + b_n \sin\omega_n x,$$

where $\omega_n = n\pi/p$ and

$$a_0 = \frac{(f,1)}{(1,1)},\ \ a_n = \frac{(f, \cos\omega_n x)}{(\cos\omega_n x, \cos\omega_n x)},\ \text{ and } b_n = \frac{(f, \sin\omega_n x)}{(\sin\omega_n x, \sin\omega_n x)}.$$

The inner product is

$$(u, v) = \int_{-p}^{p} u(x)v(x)dx.$$

The orthogonality relations are

$$(\cos\omega_n x, \cos\omega_m x) = \begin{cases} 2p & \text{if } n = m = 0 \\ p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases},$$

$$(\cos\omega_n x, \sin\omega_m x) = 0,$$

$$(\sin\omega_n x, \sin\omega_m x) = \begin{cases} p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases}.$$

0.4. Integration formulas.

$$\int x\cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

$$\int x\sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$$