MATH 3150-1, MIDTERM EXAM 2
OCTOBER 29 2008

Name: SOLUTIONS uNID: ______

Total points: 100/100.

Problem 1 (30 pts) The goal of this problem is to solve the Heat Equation with *mixed boundary conditions*

$$(1)\quad \begin{cases} u_t = u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) = 0 & \text{for } t > 0 \\ u_x(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 < x < 1 \end{cases}$$

(a) Use separation of variables to show that a general solution to (1) is

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin(\lambda_n x) \exp[-\lambda_n^2 t], \quad \text{where } \lambda_n = \frac{2n+1}{2}\pi.$$

We use ansatz $u(x,t) = X(x)T(t)$ in (1).

$XT' = X''T \Rightarrow \frac{X}{X''} = \frac{T'}{T} = -\lambda^2 < 0$ in order to have physically meaningful decaying solutions

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = X'(1) = 0 \end{cases} \qquad \begin{cases} T' + \lambda^2 T = 0 \end{cases}$$

$\rightarrow T_n(t) = b_n \exp[-\lambda_n^2 t]$

$X(x) = a\cos\lambda x + b\sin\lambda x$

$X'(x) = -a\lambda\sin\lambda x + b\lambda\cos\lambda x$

$X(0) = a = 0$

$X'(1) = +b\lambda\cos\lambda = 0$

- $b = 0$ gives trivial sol
- $\lambda = 0 \Rightarrow X(x) = \text{const} = X(0) = 0$

$\Rightarrow \cos\lambda = 0 \Rightarrow \lambda = \lambda_n = \frac{2n+1}{2}\pi$

$\Rightarrow X_n(x) = \sin\lambda_n x, \; n = 0, 1, 2, \ldots$

Product solutions $u_n(x,t) = X_n(x)T_n(t)$ satisfy (1) thus:

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin(\lambda_n x)\exp[-\lambda_n^2 t]$$

(b) Consider the inner product $(u,v) = \int_0^1 u(x)v(x)dx$. Given the orthogonality relations valid for $n = 0,1,2,\ldots$ and $m = 0,1,2,\ldots$

$$(\sin(\lambda_n x), \sin(\lambda_m x)) = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$$

show that

$$b_n = 2\int_0^1 \sin(\lambda_n x) f(x) dx, \quad \text{for } n = 0,1,2,\ldots$$

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} \sin(\lambda_n x)$$

$$(f(x), \sin\lambda_n x) = (\sin\lambda_n x, \sin\lambda_n x)\, b_n$$

$$\rightarrow \boxed{b_n = \frac{(f, \sin\lambda_n x)}{(\sin\lambda_n x, \sin\lambda_n x)} = 2\int_0^1 \sin\lambda_n x\, f(x)\, dx}$$

(c) Solve problem (1) with $f(x) = \sin(\pi x/2)$.

Here $b_n = 2\int_0^1 \sin\lambda_n x \sin\lambda_1 x = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n\neq 1 \end{cases}$

therefore.

$$\boxed{u(x,t) = \sin\left[\frac{\pi x}{2}\right] \exp\left[-\left(\frac{\pi}{2}\right)^2 t\right]}$$

Problem 2 (30 pts) Consider the 2D heat equation below, which models the temperature distribution of a square plate with all the edges dipped in an ice bath.

$$(2)\qquad \begin{cases} u_t = u_{xx} + u_{yy}, & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } t > 0 \\ u(0,y,t) = u(1,y,t) = 0 & \text{for } 0 < y < 1 \text{ and } t > 0 \quad ① \\ u(x,0,t) = u(x,1,t) = 0 & \text{for } 0 < x < 1 \text{ and } t > 0 \quad ② \\ u(x,y,0) = f(x,y) & \text{for } 0 < x < 1 \text{ and } 0 < y < 1. \end{cases}$$

(a) Show that if we assume $u(x,y,t) = X(x)Y(y)T(t)$, then separation of variables gives

$$X'' + \mu^2 X = 0, \quad X(0) = 0, \quad X(1) = 0,$$
$$Y'' + \nu^2 Y = 0, \quad Y(0) = 0, \quad Y(1) = 0,$$
$$T' + (\mu^2 + \nu^2)T = 0$$

Note: You do not need to justify the sign of the constants for each ODE.

Plugging ansatz into (2) gives:

$$XYT' = X''YT + XY''T \Rightarrow \underbrace{\frac{T'}{T}}_{\text{dep on } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{dep on } (x,y)} = -\lambda^2 \text{ (to get exp decay)}$$

Then again use separation of variables:

$$\frac{X''}{X} + \lambda^2 = -\frac{Y''}{Y} = +\nu^2 = \text{const} > 0$$

$$\Rightarrow \frac{X''}{X} = \nu^2 - \lambda^2 = -\mu^2 < 0$$

$$\Rightarrow \boxed{\begin{cases} Y'' + \nu^2 Y = 0 \\ Y(0) = Y(1) = 0 \end{cases}} \quad \text{and} \quad \boxed{\begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(1) = 0 \end{cases}}$$

B.C. come from ② since e.g. $X(x)Y(0) = 0 \ \forall x \Leftrightarrow Y(0) = 0$

B.C. coming from ①.

and $\boxed{T' + \lambda^2 T = 0}$

(b) Obtain the product solutions

$$u_{mn}(x,y,t) = B_{mn}\exp[-((m\pi)^2+(n\pi)^2)t]\sin(m\pi x)\sin(n\pi y).$$

Note: the ODEs for X and Y are very similar, so solving one of them in detail and stating the result for the other one is enough.

Solving for $X(x) = a\cos\mu x + b\sin\mu x$

$X(0) = a = 0$

$X(1) = b\sin\mu = 0 \Rightarrow \mu = \mu_m = m\pi, \quad m = 1,2,3\ldots$

$(b \neq 0)$

(otherwise trivial sol)

$\Rightarrow \boxed{X_m(x) = \sin m\pi x}$ and similarly $\boxed{Y_n(y) = \sin(n\pi y)}$

And: $\boxed{T_{mn}(t) = B_{mn}\exp\left[-((m\pi)^2+(n\pi)^2)t\right]}$

Thus the product solutions are:

$u_{mn}(x,y,t) = X_m(x)\,Y_n(y)\,T_{mn}(t)$

$= B_{mn}\exp\left[-((m\pi)^2+(n\pi)^2)t\right]\sin(m\pi x)\sin(n\pi y)$

(c) Write down the general form of a solution $u(x,y,t)$ to (2), and use the formulas at the end of the exam to express B_{mn} in terms of $f(x,y)$.

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\exp\left[-((m\pi)^2+(n\pi)^2)t\right]\sin(m\pi x)\sin(n\pi y)$$

$$u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin(m\pi x)\sin(n\pi y)$$

$\Rightarrow (f, \sin(m\pi x)\sin(n\pi y)) = B_{mn}(\sin m\pi x\sin n\pi y, \sin m\pi x\sin n\pi y)$

$\Rightarrow \boxed{B_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy}$

where we use orthogonality relations for such functions.

(d) Using the formulas at end of this exam and that

$$\int_0^1 x(1-x)\sin(m\pi x)dx = \frac{2((-1)^m-1)}{\pi^3 m^3},$$

find the coefficients B_{mn} in the double sine series

$$x(1-x)y(1-y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin(m\pi x)\sin(n\pi y).$$

$$B_{mn} = \int_0^1\int_0^1 x(1-x)y(1-y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy = \left[\int_0^1 x(1-x)\sin(m\pi x)\,dx\right] \times$$

$$\times\left[\int_0^1 y(1-y)\sin(n\pi y)\right]$$

$$= \boxed{\frac{4((-1)^n-1)((-1)^n-1)}{\pi^6 m^3 n^3}}$$

(e) Solve the 2D heat equation (2) with $f(x,y) = x(1-x)y(1-y)$.

$$\boxed{u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{mn}\sin(m\pi x)\sin(n\pi y)\exp\left[-((m\pi)^2+(n\pi)^2)t\right]}$$

where B_{mn} is given as above.

Problem 3 (30 pts) Consider a circular membrane of radius 2 with radially symmetric initial shape $f(r)$ and zero initial velocity. Then the displacement from equilibrium satisfies the 2D Wave equation

$$(3)\quad \begin{cases} u_{tt} = \Delta u, & \text{for } 0 < r < 2 \text{ and } t > 0 \\ u(r,0) = f(r), & \text{for } 0 < r < 2 \\ u_t(r,0) = 0, & \text{for } 0 < r < 2 \\ u(2,t) = 0, & \text{for } t > 0. \end{cases}$$

Carrying out the method of separation of variables gives that a general solution to (3) has the form

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\frac{\alpha_n}{2} r) \cos(\frac{\alpha_n}{2} t),$$

with α_n being the n–th positive zero of the first kind zeroth order Bessel function $J_0(r)$.

(a) Use the initial conditions and the orthogonality relations for Bessel functions (see end of this exam) to show that

$$A_n = \frac{1}{2J_1^2(\alpha_n)} \int_0^2 f(r) J_0(\frac{\alpha_n}{2} r) r\, dr.$$

Using B.C. : $u(r,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{2} r\right) = f(r)$

Thus $A_n \left(J_0\left(\frac{\alpha_n}{2} r\right), J_0\left(\frac{\alpha_n}{a} r\right)\right) = \left(f(r), J_0\left(\frac{\alpha_n}{a} r\right)\right)$

$A_n \left(\frac{4}{2} J_1^2(\alpha_{0n})\right) = \int_0^1 f(r) J_0\left(\frac{\alpha_n}{a} r\right) r\, dr$

$\Rightarrow \boxed{A_n = \frac{1}{2 J_1^2(\alpha_{0n})} \int_0^1 r\, dr\, f(r) J_0\left(\frac{\alpha_n r}{2}\right)}$

(b) Solve (3) with $f(r) = 4 - r^2$. **Hint:** Use the last integration formula in §0.3.

From §0.3:

$$\int_0^2 (4-r^2)\, r J_0\left(\frac{\alpha_n}{2} r\right) dr = \frac{2 \cdot 2^4}{\alpha_n^2} J_2(\alpha_n)$$

Then $\boxed{A_n = \frac{1}{2J_1^2(\alpha_n)} \frac{2 \cdot 2^4}{\alpha_n^2} J_2(\alpha_n) = \frac{16 J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)}}$

where $\alpha_n = \alpha_{0n}$ = nth pos zero of Bessel fun.

Thus: $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{2} r\right) \cos\left(\frac{\alpha_n}{2} t\right)}$

Problem 4 (10 pts) Recall that the Laplacian in polar coordinates is:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Determine whether the function $f(x, y) = \ln(x^2 + y^2)$ satisfies Laplace's equation $\Delta u = 0$.

$f(x,y) = \ln r^2 = 2 \ln r$ $\quad$ f does not depend on θ

$\Delta f = -\frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} 0 = 0$

Using $(\ln r)' = \frac{1}{r}$

$(\ln r)' = \frac{2}{r}$

Thus f solves Laplace eq.