MATH 3150-1, PRACTICE MIDTERM EXAM 2
OCTOBER 24 2008
SOLUTIONS

Total points: 100/100.

Problem 1 (30 pts) The goal of this problem is to solve the Heat Equation with *mixed boundary conditions*

$$(1)\qquad \begin{cases} u_t = 3u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u_x(0,t) = 0 & \text{for } t > 0 \\ u(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 < x < 1 \end{cases}$$

(a) Use separation of variables to show that a general solution to (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x) \exp[-3\lambda_n^2 t], \quad \text{where } \lambda_n = \frac{2n+1}{2}\pi.$$

We take $u(x,t) = X(x)T(t)$ and plug into (1) to get:

$XT' = 3X''T$

$\frac{T'}{3T} = \frac{X''}{X} = -\lambda^2$ (negative constant to reproduce physical decay behavior of temp)

$\Rightarrow \quad T' - \lambda^2 3T = 0 \quad \& \quad \begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) = X(1) = 0 \end{cases}$

$\Rightarrow \quad T(t) = a\exp[-3\lambda^2 t]$

Thus $X(x) = a\cos\lambda t + b\sin\lambda t$

B.C. $\Rightarrow X'(x) = -a\lambda\sin\lambda t + b\lambda\cos\lambda t$

$X'(0) = b\lambda = 0 \Rightarrow b = 0$ or $\lambda = 0$

if $b=0$: $X(1) = a\cos\lambda = 0 \Rightarrow \lambda = \lambda_n = \frac{2n+1}{2}\pi, n = 0, 1, \ldots$

if $\lambda = 0$: $X(x) = a$ and $X(1) = 0 \Rightarrow$ trivial sol.

$\Rightarrow \boxed{\begin{array}{l} X_n(x) = \cos(\lambda_n x) \\ T_n(t) = a_n \exp[-\lambda_n^2 3t] \end{array}}$ for $n = 0, 1, \ldots$ $\Rightarrow$ product solutions $u_n(x,t) = a_n\cos(\lambda_n x)\exp[-3\lambda_n^2 t]$

and finally we get general form by supperposition:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x)\exp[-3\lambda_n^2 t]$$

(b) Consider the inner product $(u, v) = \int_0^1 u(x)v(x)dx$. Given the orthogonality relations valid for $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$

$$(\cos(\lambda_n x), \cos(\lambda_m x)) = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$$

show that

$$a_n = 2\int_0^1 \cos(\lambda_n x) f(x)dx, \quad \text{for } n = 0, 1, 2, \ldots$$

at $t=0$: $u(x,0) = f(x) = \sum_{m=0}^{\infty} a_m \cos \lambda_m x$

$\Rightarrow \quad (f, \cos\lambda_n x) = a_n (\cos\lambda_n, \cos\lambda_n)$ thanks to orthogonality relations.

$\Rightarrow \quad a_n = \dfrac{(f, \cos\lambda_n x)}{(\cos\lambda_n x, \cos\lambda_n x)} = 2\int_0^1 f(x)\cos\lambda_n x \, dx$

(c) Solve problem (1) with $f(x) = \cos(3\pi x/2) + 2\cos(7\pi x/2)$.

With orthogonality relations we see that

$$a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\boxed{u(x,t) = \cos\left(\frac{3\pi}{2}x\right)\exp\left[-3\left(\frac{3\pi}{2}\right)^2 t\right] + 2\cos\left(\frac{7\pi}{2}x\right)\exp\left[-3\left(\frac{7\pi}{2}\right)^2 t\right].}$$

Problem 2 (30 pts) Consider the 2D Laplace equation below, which models the steady state temperature distribution of a square plate where the right and left sides are kept in an ice bath and the bottom and top sides have prescribed temperatures $f_1(x)$ and $f_2(x)$ respectively.

$$(2)\quad \begin{cases} u_{xx} + u_{yy} = 0, & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ u(0,y) = u(1,y) = 0, & \text{for } 0 < y < 1 \\ u(x,0) = f_1(x), & \text{for } 0 < x < 1 \\ u(x,1) = f_2(x), & \text{for } 0 < x < 1. \end{cases}$$

(a) Explain why it is possible to decompose (2) into the two subproblems below (the x and y below are implicitly in $(0,1)$).

$$(\text{P1}) \begin{cases} v_{xx} + v_{yy} = 0, \\ v(0,y) = v(1,y) = 0, \\ v(x,0) = f_1(x), \\ v(x,1) = 0 \end{cases} \qquad (\text{P2}) \begin{cases} w_{xx} + w_{yy} = 0, \\ w(0,y) = w(1,y) = 0, \\ w(x,0) = 0, \\ w(x,1) = f_2(x) \end{cases}$$

(2) is a homog linear equation. Thus if v solves (P1) and w solves (P2) then $u = v + w$ solves (2).

(b) Show that if we assume that the solution to (P2) is $w(x,y) = X(x)Y(y)$, then separation of variables gives

$$X'' + kX = 0, \quad X(0) = 0, \quad X(1) = 0$$
$$Y'' - kY = 0, \quad Y(0) = 0$$

Plugging $w(x,y)$ into (2) gives:

$$X''Y + XY'' = 0 \iff \frac{X''}{X} = -\frac{Y''}{Y} = k \text{ constant}$$

$$\begin{cases} X'' + kX = 0 \\ X(0) = X(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} Y'' - kY = 0 \\ Y(0) = 0 \end{cases}$$

comes from B.C. at right and left sides

comes from B.C. at bottom side

(c) Assuming $k = \mu^2 > 0$, obtain the product solutions to (P2)

$$w_n(x,y) = B_n \sin(n\pi x)\sinh(n\pi y)$$

Solving for X:

$X(x) = a\cos\mu x + b\sin\mu x$

$X(0) = a = 0$

$X(1) = b\sin\mu = 0$

$\Rightarrow \mu = \mu_n = n\pi$

$\Rightarrow X_n(x) = \sin(n\pi x)$

(for $n = 1, 2, \ldots$)

Solving for Y:

$Y_n(y) = a\cosh\mu_n y + b\sinh\mu_n y$

$Y_n(0) = 0 = a$

$\Rightarrow Y_n(y) = b_n \sinh\mu_n y = b_n\sinh(n\pi y)$

product solutions:

$\Rightarrow w_n(x,y) = b_n \sin(n\pi x)\sinh(n\pi y)$

(for $n = 1, 2, \ldots$).

(d) Write down the general form of a solution to (P2), and use the formulas at the end of the exam to express B_n in terms of $f_2(x)$.

$$w(x,y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh(n\pi y)$$

B.C.:

$$w(x,1) = f_2(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh n\pi$$

$$\Rightarrow B_n \sinh n\pi = \frac{(f_2, \sin n\pi x)}{(\sin n\pi x, \sin n\pi x)} = 2\int_0^1 f_2(x)\sin n\pi x \, dx$$

$$\Rightarrow \boxed{B_n = \frac{2}{\sinh n\pi}\int_0^1 f_2(x)\sin n\pi x \, dx}$$

(e) In a similar way it is possible to obtain the product solutions to (P1),

$$v_n(x,y) = A_n \sin(n\pi x)\sinh(n\pi(1-y)).$$

Write down the general form of a solution to (P1) and give an expression for A_n in terms of $f_1(x)$.

$$v(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi(1-y)$$

B.C. $f_1(x) = v(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi$

$\Rightarrow$ similarly $\boxed{A_n = \frac{2}{\sinh n\pi}\int_0^1 \sin n\pi x \, f_1(x)\,dx}$

(f) Solve (2) with $f_1(x) = 100$ and $f_2(x) = 100x(1-x)$. You may use the identity below (valid for $n = 1, 2, \ldots$):

$$\int_0^1 x(1-x)\sin(n\pi x)dx = \frac{2((-1)^n - 1)}{\pi^3 n^3}.$$

With: $A_n = \frac{2}{\sinh n\pi}\int_0^1 \sin n\pi x \, 100\,dx$

$$= \frac{200}{\sinh n\pi}\frac{(1-(-1)^n)}{n\pi}$$

$$B_n = \frac{2}{\sinh n\pi}\int_0^1 \sin n\pi x \, 100x(1-x)\,dx$$

$$= \frac{400}{\sinh n\pi}\frac{((-1)^n - 1)}{\pi^3 n^3}$$

The solution to (2) is:

$$u(x,y) = v(x,y) + w(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi(1-y)$$

$$+ \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y$$

Problem 3 (30 pts) Consider a circular plate of radius 1 with initial temperature distribution of the form $f(r,\theta) = g(r)\cos 2\theta$ and where the outer rim of the plate is kept in an ice bath. The temperature distribution $u(r,\theta,t)$ satisfies the 2D Heat equation

$$(3) \qquad \begin{cases} u_t = \Delta u & \text{for } 0 < r < 1,\ 0 \le \theta \le 2\pi \text{ and } t > 0 \\ u(r,\theta,0) = f(r,\theta) & \text{for } 0 < r < 1 \text{ and } 0 \le \theta \le 2\pi \\ u(1,\theta,t) = 0 & \text{for } 0 \le \theta \le 2\pi \text{ and } t > 0 \end{cases}$$

Because the initial temperature distribution is a multiple of $\cos 2\theta$, the solution can be shown to be

$$u(r,\theta,t) = \sum_{n=1}^{\infty} a_{2n} J_2(\alpha_{2n} r)\cos 2\theta \exp[-\alpha_{2n}^2 t].$$

where α_{2n} denotes the n–th zero of the Bessel function of the first kind of order 2, and

$$a_{2n} = \frac{2}{\pi J_{2+1}^2(\alpha_{2n})} \int_0^1 \int_0^{2\pi} f(r,\theta) J_2(\alpha_{2n} r)\cos 2\theta \, d\theta \, r dr \quad \text{for } n = 1, 2, \ldots$$

(a) Solve (3) with the initial temperatures

$$f_1(r,\theta) = J_2(\alpha_{2,1} r)\cos 2\theta \quad \text{and} \quad f_2(r,\theta) = J_2(\alpha_{2,2} r)\cos 2\theta.$$

For $f_1(r,\theta)$:

$$a_{2n} = \frac{2}{J_3^2(\alpha_n)} \left[\int_0^1 J_2(\alpha_{2n} r) J_2(\alpha_{2,1} r)\, r dr\right] \left[\frac{1}{\pi} \underbrace{\int_0^{2\pi} \cos^2 2\theta \, d\theta}_{=1}\right] = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow u_1(r,\theta,t) = J_2(\alpha_{21} r)\cos 2\theta \exp[-\alpha_{21}^2 t]$$

$$u_2(r,\theta,t) = J_2(\alpha_{22} r)\cos 2\theta \exp(-\alpha_{22}^2 t)$$

(b) The steady state temperature distribution is $u = 0$. Of the initial temperatures $f_1(r,\theta)$ and $f_2(r,\theta)$, which decays faster to the steady state? Justify your answer.

$\alpha_{21} < \alpha_{22}$ thus $u_2 \to 0$ faster than u_1.

Problem 4 (10 pts) Recall that the Laplacian in spherical coordinates is:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right).$$

Determine whether the function $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies Laplace's equation $\Delta u = 0$.

In spherical coordinates:

$$f(x,y,z) = f(r) = \frac{1}{r}$$

$$\frac{\partial f}{\partial r} = -\frac{1}{r^2}$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{2}{r^3}$$

$$\Rightarrow \Delta f = \frac{2}{r^3} + \frac{2}{r}\left(-\frac{1}{r^2}\right) + \frac{1}{r^2}\underbrace{\left(\quad\right)}_{=0 \text{ since } f \text{ does not depend on } \theta \text{ or } \varphi}$$

$$= 0$$

$\Rightarrow f$ satisfies Laplace eq.