MATH 3150-1, PRACTICE MIDTERM EXAM 2
OCTOBER 24 2008

Total points: 100/100.

Problem 1 (30 pts) The goal of this problem is to solve the Heat Equation with *mixed boundary conditions*

$$(1)\qquad \begin{cases} u_t = 3u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u_x(0,t) = 0 & \text{for } t > 0 \\ u(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 < x < 1 \end{cases}$$

(a) Use separation of variables to show that a general solution to (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x) \exp[-3\lambda_n^2 t], \quad \text{where } \lambda_n = \frac{2n+1}{2}\pi.$$

(b) Consider the inner product $(u,v) = \int_0^1 u(x)v(x)dx$. Given the orthogonality relations valid for $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$

$$(\cos(\lambda_n x), \cos(\lambda_m x)) = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$$

show that

$$a_n = 2\int_0^1 \cos(\lambda_n x) f(x) dx, \quad \text{for } n = 0, 1, 2, \ldots$$

(c) Solve problem (1) with $f(x) = \cos(3\pi x/2) + 2\cos(7\pi x/2)$.

Problem 2 (30 pts) Consider the 2D Laplace equation below, which models the steady state temperature distribution of a square plate where the right and left sides are kept in an ice bath and the bottom and top sides have prescribed temperatures $f_1(x)$ and $f_2(x)$ respectively.

$$(2)\qquad \begin{cases} u_{xx} + u_{yy} = 0, & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ u(0,y) = u(1,y) = 0, & \text{for } 0 < y < 1 \\ u(x,0) = f_1(x), & \text{for } 0 < x < 1 \\ u(x,1) = f_2(x), & \text{for } 0 < x < 1. \end{cases}$$

(a) Explain why it is possible to decompose (2) into the two subproblems below (the x and y below are implicitly in $(0,1)$).

$$(\text{P1}) \begin{cases} v_{xx} + v_{yy} = 0, \\ v(0,y) = v(1,y) = 0, \\ v(x,0) = f_1(x), \\ v(x,1) = 0 \end{cases} \qquad (\text{P2}) \begin{cases} w_{xx} + w_{yy} = 0, \\ w(0,y) = w(1,y) = 0, \\ w(x,0) = 0, \\ w(x,1) = f_2(x) \end{cases}$$

(b) Show that if we assume that the solution to (P2) is $w(x,y) = X(x)Y(y)$, then separation of variables gives

$$X'' + kX = 0, \quad X(0) = 0, \quad X(1) = 0$$
$$Y'' - kY = 0, \quad Y(0) = 0$$

(c) Assuming $k = \mu^2 > 0$, obtain the product solutions to (P2)

$$w_n(x,y) = B_n \sin(n\pi x) \sinh(n\pi y)$$

(d) Write down the general form of a solution to (P2), and use the formulas at the end of the exam to express B_n in terms of $f_2(x)$.

(e) In a similar way it is possible to obtain the product solutions to (P1),

$$v_n(x,y) = A_n \sin(n\pi x) \sinh(n\pi(1-y)).$$

Write down the general form of a solution to (P1) and give an expression for A_n in terms of $f_1(x)$.

(f) Solve (2) with $f_1(x) = 100$ and $f_2(x) = 100x(1-x)$. You may use the identity below (valid for $n = 1, 2, \ldots$):

$$\int_0^1 x(1-x)\sin(n\pi x)dx = \frac{2((-1)^n - 1)}{\pi^3 n^3}.$$

Problem 3 (30 pts) Consider a circular plate of radius 1 with initial temperature distribution of the form $f(r,\theta) = g(r)\cos 2\theta$ and where the outer rim of the plate is kept in an ice bath. The temperature distribution $u(r,\theta,t)$ satisfies the 2D Heat equation

$$(3) \qquad \begin{cases} u_t = \Delta u & \text{for } 0 < r < 1,\ 0 \le \theta \le 2\pi \text{ and } t > 0 \\ u(r,\theta,0) = f(r,\theta) & \text{for } 0 < r < 1 \text{ and } 0 \le \theta \le 2\pi \\ u(1,\theta,t) = 0 & \text{for } 0 \le \theta \le 2\pi \text{ and } t > 0 \end{cases}$$

Because the initial temperature distribution is a multiple of $\cos 2\theta$, the solution can be shown to be

$$u(r,\theta,t) = \sum_{n=1}^{\infty} a_{2n} J_2(\alpha_{2n} r) \cos 2\theta \exp[-\alpha_{2n}^2 t].$$

where α_{2n} denotes the $n-$th zero of the Bessel function of the first kind of order 2, and

$$a_{2n} = \frac{2}{\pi J_{2+1}^2(\alpha_{2n})} \int_0^1 \int_0^{2\pi} f(r,\theta) J_2(\alpha_{2n} r) \cos 2\theta \, d\theta \, r dr \quad \text{for } n = 1, 2, \ldots$$

(a) Solve (3) with the initial temperatures

$$f_1(r,\theta) = J_2(\alpha_{2,1} r)\cos 2\theta \quad \text{and} \quad f_2(r,\theta) = J_2(\alpha_{2,2} r)\cos 2\theta.$$

(b) The steady state temperature distribution is $u = 0$. Of the initial temperatures $f_1(r,\theta)$ and $f_2(r,\theta)$, which decays faster to the steady state? Justify your answer.

Problem 4 (10 pts) Recall that the Laplacian in spherical coordinates is:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right).$$

Determine whether the function $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies Laplace's equation $\Delta u = 0$.

Some useful formulas

0.1. **Orthogonality relations for sine series.** With the inner product $(u, v) = \int_0^a u(x)v(x)dx$, we have for all m, n non-zero integers,

$$\left(\sin\left(\frac{m\pi}{a}x\right), \sin\left(\frac{n\pi}{a}x\right)\right) = \begin{cases} \frac{2}{a} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

0.2. **Hyperbolic trigonometry.**

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x$$
$$\cosh^2 x - \sinh^2 y = 1, \qquad \sinh 0 = 0$$

0.3. **Bessel functions.** The following identities are valid for $p \geq 0$ and $n = 0, 1, \ldots$.

$$\int J_1(r)dr = -J_0(r) + C \quad \text{and} \quad \int r^{p+1}J_p(r)dr = r^{p+1}J_{p+1}(r) + C$$

0.4. **Orthogonality relations for Bessel functions.** Let $a > 0$ and $m \geq 0$ be fixed. Denote with α_{mn} the $n-$th positive zero of the Bessel function of the first kind of order m. With the inner product

$$(u, v) = \int_0^a u(r)v(r)r\,dr$$

we have for all j, k non-zero integers,

$$\left(J_m(\frac{\alpha_{mj}}{a}r), J_m(\frac{\alpha_{mk}}{a}r)\right) = \begin{cases} \frac{a^2}{2}J_{m+1}^2(\alpha_{mj}) & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$