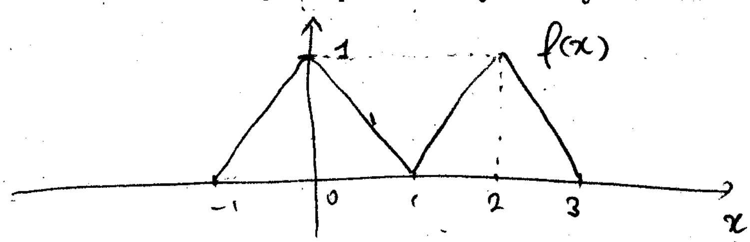
MATH 3150-1, MIDTERM EXAM 1
SEPTEMBER 24 2008

Name: SOLUTIONS uNID: ____________

Total points: 105/100.
Problem 1 (25 pts) Consider the 2−periodic function

$$f(x) = \begin{cases} 1+x & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \end{cases}$$

(a) Sketch $f(x)$ for $x \in [-1,3]$. Carefully label your axis.



(b) Is f continuous? Piecewise continuous? Piecewise smooth? f is continuous and thus it piecewise cont as well.

Since $f'(x) = \begin{cases} 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \end{cases}$, f' is piecewise cont $\Rightarrow$ f is piecewise smooth

(c) Find the Fourier series of f. (**Hint:** Use integration by parts)

Since f is an even function: $\boxed{b_n = \int_{-1}^{1} \underbrace{f(x)\sin n\pi x}_{odd}\, dx = 0}$, $n \geq 1$

$\boxed{a_0 = \int_0^1 f(x)\,dx = \frac{1}{2}}$

$$a_n = 2\int_0^1 \underbrace{f(x)}_{=(1-x)} \cos n\pi x\, dx = 2(1-x)\frac{1}{n\pi}\sin n\pi x \Big|_0^1 + 2\int_0^1 \frac{1}{n\pi}\sin n\pi x\, dx$$

$$= -\frac{2}{(n\pi)^2}\cos n\pi x\Big|_0^1 = -\frac{2}{(n\pi)^2}((-1)^n - 1) = \begin{cases} \frac{4}{(n\pi)^2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

thus $\boxed{f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{((2k+1)\pi)^2} \cos(2k+1)\pi x}$

Problem 2 (15 pts) Decide whether the following partial differential equations are linear or non-linear and if linear, whether they are homogeneous or non-homogeneous. Determine the order of the differential equation.

(a) $\begin{cases} u_{xx} + u_{xy} = 2u \\ u_x(0, y) = 0 \end{cases}$ 2nd order linear homogeneous

(c) $\begin{cases} u_{xx} - u_t = \sin(x+t) \\ u_t(x, 0) = 2 \end{cases}$ 2nd order linear non-homog

(b) $\begin{cases} u_x + xu_{xy} = 2 \\ u(x, 0) = u(x, 1) = 0 \end{cases}$ 2rd order linear non-homogeneous

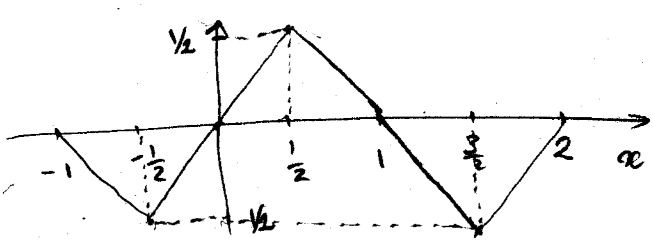
(d) $\begin{cases} u_{xx} + u^2 = u_t \\ u(x, 0) = 1 \\ u(x, 1) = 0 \end{cases}$ 2nd order non-linear

Problem 3 (25 pts) Consider the function $f(x)$ defined on $[0,1]$ by

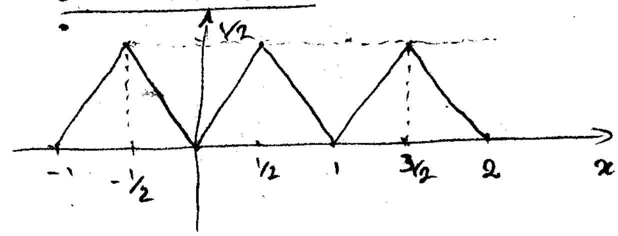
$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1-x & \text{if } 1/2 < x \le 1 \end{cases}$$

(a) Sketch the odd and even 2−periodic extensions of $f(x)$. Carefully label your axis.

odd extension



even extension



(b) Calculate the sine series expansion of $f(x)$.

Hint: $f(x) = \sum_{k=0}^{\infty} \frac{4(-1)^k}{((2k+1)\pi)^2} \sin((2k+1)\pi x)$, for $x \in [0,1]$. Use integration by parts.

$$f(x) = \sum_{n\geq 1}^{\infty} b_n \sin n\pi x \, dx$$

where $\boxed{b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx}$

$$= 2\int_0^{1/2} x \sin n\pi x \, dx + 2\int_{\frac{1}{2}}^1 (1-x)\sin n\pi x \, dx$$

$$= -\frac{2}{n\pi} x \cos n\pi x \Big|_0^{\frac{1}{2}} + \frac{2}{n\pi}\int_0^{1/2} \cos n\pi x \, dx$$

$$-\frac{1}{n\pi}(1-x)\cos n\pi x \Big|_{\frac{1}{2}}^1 - \frac{2}{n\pi}\int_{\frac{1}{2}}^1 \cos n\pi x \, dx$$

$$= -\frac{1}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{(n\pi)^2}\sin n\pi x \Big|_0^{1/2}$$

$$+ \frac{1}{n\pi}\cos\frac{n\pi}{2} - \frac{2}{(n\pi)^2}\sin n\pi x \Big|_{1/2}^1$$

$$= \boxed{\frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}}$$

n mod 4	$\sin\frac{n\pi}{2}$
0	0
1	1
2	0
3	-1

$$\Rightarrow \boxed{f(x) = \sum_{k=0}^{\infty} \frac{4}{((2k+1)\pi)^2} (-1)^k \sin(2k+1)\pi x}$$

(c) Solve the one dimensional wave equation

$$\begin{cases} u_{tt} = 4u_{xx}, & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = f(x), & x \in [0,1] \\ u_t(x,0) = 0, & x \in [0,1] \end{cases}$$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos 2n\pi t$, where b_n is def as in (b).

(d) Solve the one dimensional wave equation

$$\begin{cases} u_{tt} = 4u_{xx}, & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = 0, & x \in [0,1] \\ u_t(x,0) = f(x), & x \in [0,1] \end{cases}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n^* \sin n\pi x \sin 2n\pi t}$$

$u_t(x,t) = \sum_{n=1}^{\infty} b_n^* (2n\pi) \sin n\pi x \cos 2n\pi t$

$u_t(x,0) = f(x) \Rightarrow \boxed{b_n^* = \frac{1}{2n\pi} b_n}$, where b_n is given as in (b).

Problem 4 (25 pts) Consider the one dimensional wave equation

$$\begin{cases} u_{tt} = u_{xx}, & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = \sin(\pi x), & x \in (0,1) \\ u_t(x,0) = 0, & x \in (0,1) \end{cases}$$

(a) Solve this differential equation using Fourier series.

The Fourier serie of $f(x) = \sin\pi x$ are itself i.e. there is a single non-zero term with $b_1 = \int_{-1}^{1} (\sin\pi x)^2 dx = 1$

$b_n = 0, \; n \neq 0$

Thus $\boxed{u(x,t) = \sin\pi x \cos\pi t}$

(b) Solve this differential equation using D'Alembert's method.

We have zero initial velocity, therefore

$$\boxed{u(x,t) = \frac{1}{2}\left[\sin(\pi(x+t)) + \sin(\pi(x-t))\right].}$$

(c) Show that the solutions from (a) and (b) are equal.

We use trig formulas to transform (b):

$$u(x,t) = \frac{1}{2}\left[\sin(\pi x)\cos(\pi t) + \sin(\pi t)\cos(\pi x)\right]$$
$$+ \frac{1}{2}\left[\sin(\pi x)\cos(\pi t) - \sin(\pi t)\cos(\pi x)\right]$$

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$$= \sin(\pi x)\cos(\pi t).$$

Problem 5 (15 pts) Solve the heat equation

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = f(x), & \text{defined as in Problem 3 for } x \in [0,1]. \end{cases}$$

We have:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \exp\left[-(n\pi)^2 t\right]$$

with b_n defined as in problem 3b.